Digital Signal Processing – Chapter 5

The Fourier Transform

CES Engineering

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Fourier Transform Definition

$$x(t) \quad \Leftrightarrow \quad X(\Omega)$$

where the signal $x(t)$ is transformed into a function $X(\Omega)$ in the frequency domain by the

$$\text{Fourier transform:} \qquad X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

while $X(\Omega)$ is transformed into a signal $x(t)$ in the time domain by the

$$\text{Inverse Fourier transform:} \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$

Fourier Transforms from Laplace Transforms

- If the region of convergence (ROC) of the Laplace transform *X(s)* contains the $j\Omega$ axis, so that $X(s)$ can be defined for s= D $j\Omega$, then:

$$\boxed{\begin{aligned}\mathcal{F}[x(t)] &= \mathcal{L}[x(t)]|_{s=j\Omega} = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \\ &= X(s)\,|_{s=j\Omega}\end{aligned}}$$

Table 5.2 Fourier Transform Pairs

	Function of Time	Function of Ω
1	$\delta(t)$	1
2	$\delta(t-\tau)$	$e^{-j\Omega\tau}$
3	$u(t)$	$\frac{1}{j\Omega} + \pi\delta(\Omega)$
4	$u(-t)$	$\frac{-1}{j\Omega} + \pi\delta(\Omega)$
5	$\mathrm{sgn}(t) = 2[u(t) - 0.5]$	$\frac{2}{j\Omega}$
6	$A, \quad -\infty < t < \infty$	$2\pi A\delta(\Omega)$
7	$Ae^{-at}u(t), \ a > 0$	$\frac{A}{j\Omega+a}$
8	$Ate^{-at}u(t), \ a > 0$	$\frac{A}{(j\Omega+a)^2}$
9	$e^{-a\|t\|}, \ a > 0$	$\frac{2a}{a^2+\Omega^2}$
10	$\cos(\Omega_0 t), \quad -\infty < t < \infty$	$\pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$
11	$\sin(\Omega_0 t), \quad -\infty < t < \infty$	$-j\pi[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$
12	$A[u(t+\tau) - u(t-\tau)], \ \tau > 0$	$2A\tau\frac{\sin(\Omega\tau)}{\Omega\tau}$
13	$\frac{\sin(\Omega_0 t)}{\pi t}$	$u(\Omega + \Omega_0) - u(\Omega - \Omega_0)$
14	$x(t)\cos(\Omega_0 t)$	$0.5[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$

Linearity

- Fourier transform is a linear operator
- Superposition holds

If $\mathcal{F}[x(t)] = X(\Omega)$ and $\mathcal{F}[y(t)] = Y(\Omega)$, for constants α and β, we have that

$$\begin{aligned}\mathcal{F}[\alpha x(t) + \beta y(t)] &= \alpha \mathcal{F}[x(t)] + \beta \mathcal{F}[y(t)] \\ &= \alpha X(\Omega) + \beta Y(\Omega)\end{aligned}$$

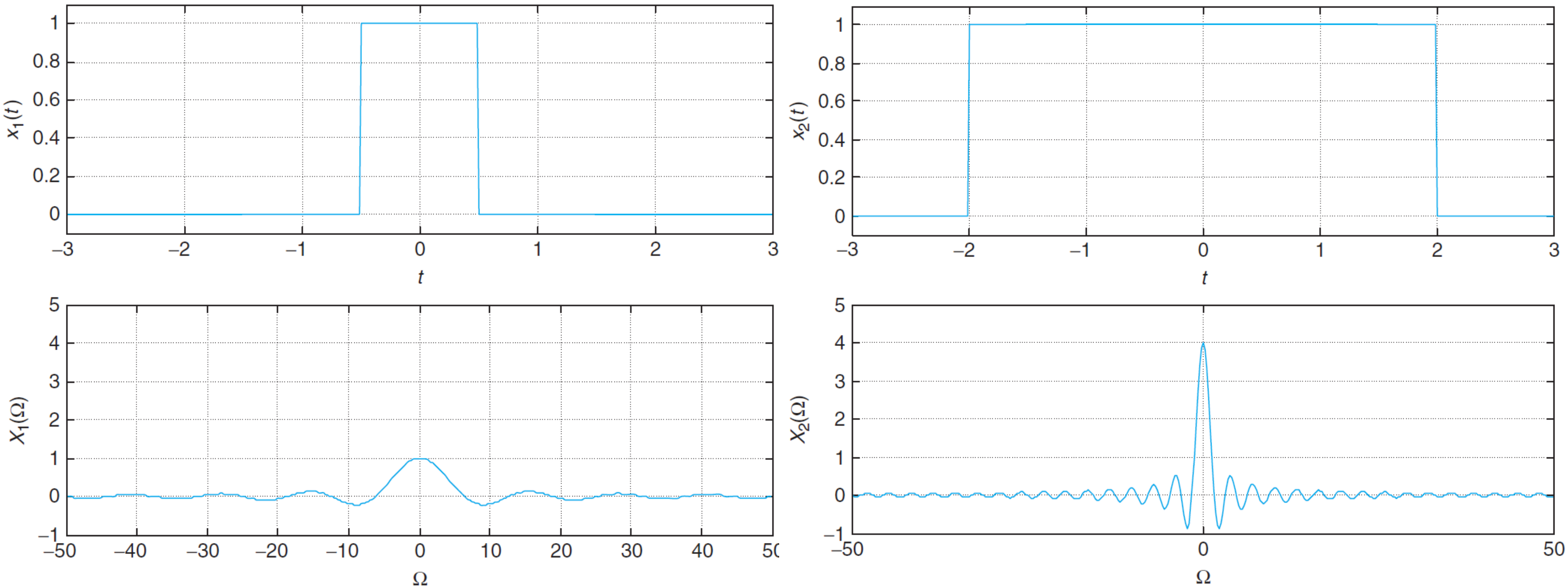
Inverse Proportionality of Time and Frequency

- Support of $X(\Omega)$ is inversely proportional to support of $x(t)$
- If $x(t)$ has a Fourier transform $X(\Omega)$ and $\alpha \neq 0$ is a real number, then $x(\alpha t)$ is:
  - Contracted $(\alpha > 1)$,
  - Contracted and reflected $(\alpha < -1)$,
  - Expanded $(0 < \alpha < 1)$,
  - Expanded and reflected $(-1 < \alpha < 0)$, or
  - Simply reflected $(\alpha = -1)$
- Then,

$$\boxed{x(\alpha t) \quad \Leftrightarrow \quad \frac{1}{|\alpha|} X\left(\frac{\Omega}{\alpha}\right)}$$

Inverse Proportionality of Time and Frequency - Example

- Fourier transform of 2 pulses of different width
  - 4-times wider pulse have 4-times narrower Fourier transform



Duality

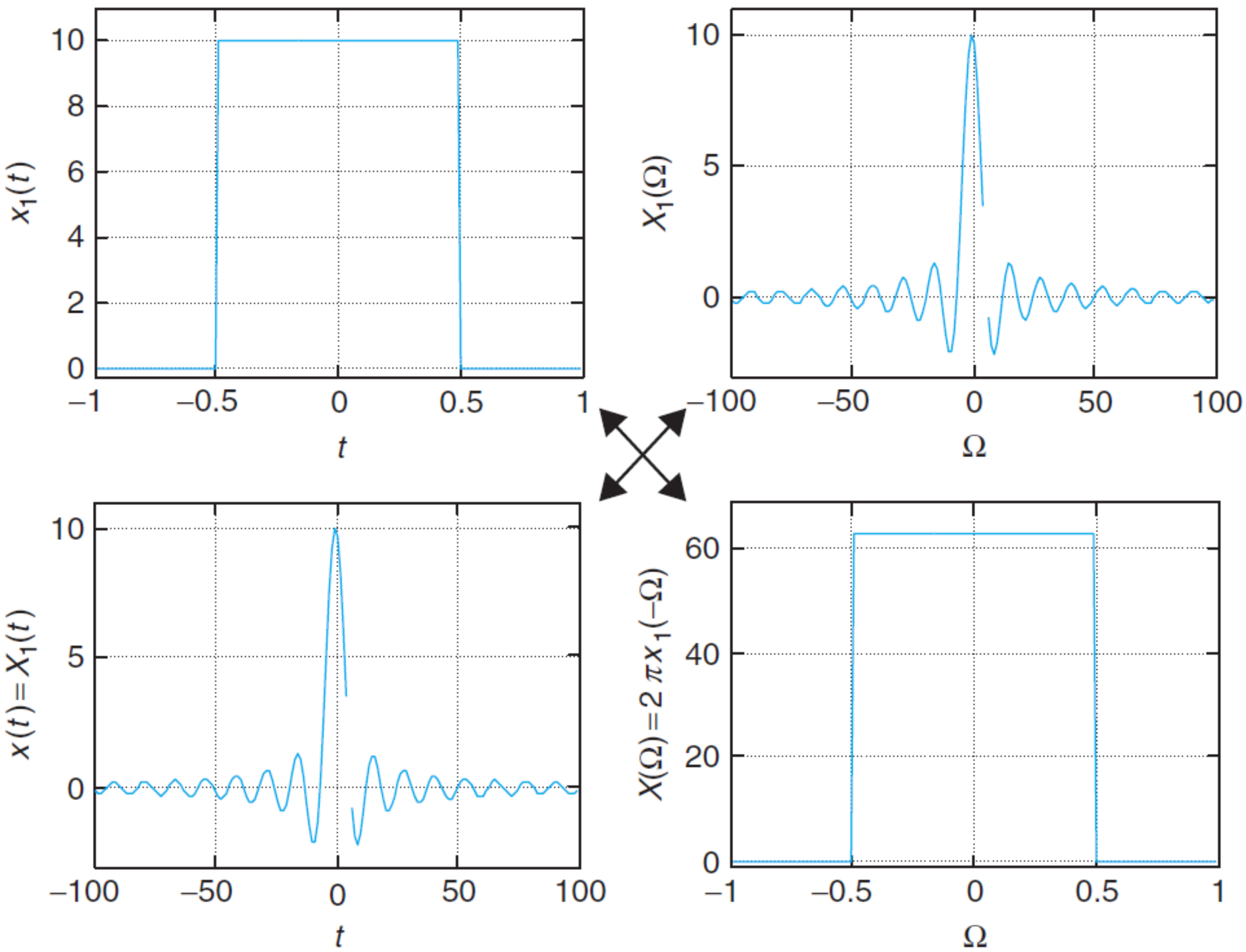
- By interchanging the frequency and the time variables in the definitions of the direct and the inverse Fourier transform similar equations are obtained
- Thus, the direct and the inverse Fourier transforms are dual

$$x(t) \quad \Leftrightarrow \quad X(\Omega)$$

$$\Downarrow$$

$$\boxed{X(t) \quad \Leftrightarrow \quad 2\pi x(-\Omega)}$$

Duality: Example



Signal Modulation

- Frequency shift: If $X(\Omega)$ is the Fourier transform of $x(t)$, then we have the pair

$$x(t)e^{j\Omega_0 t} \quad \Leftrightarrow \quad X(\Omega - \Omega_0)$$

- Modulation: The Fourier transform of the modulated signal

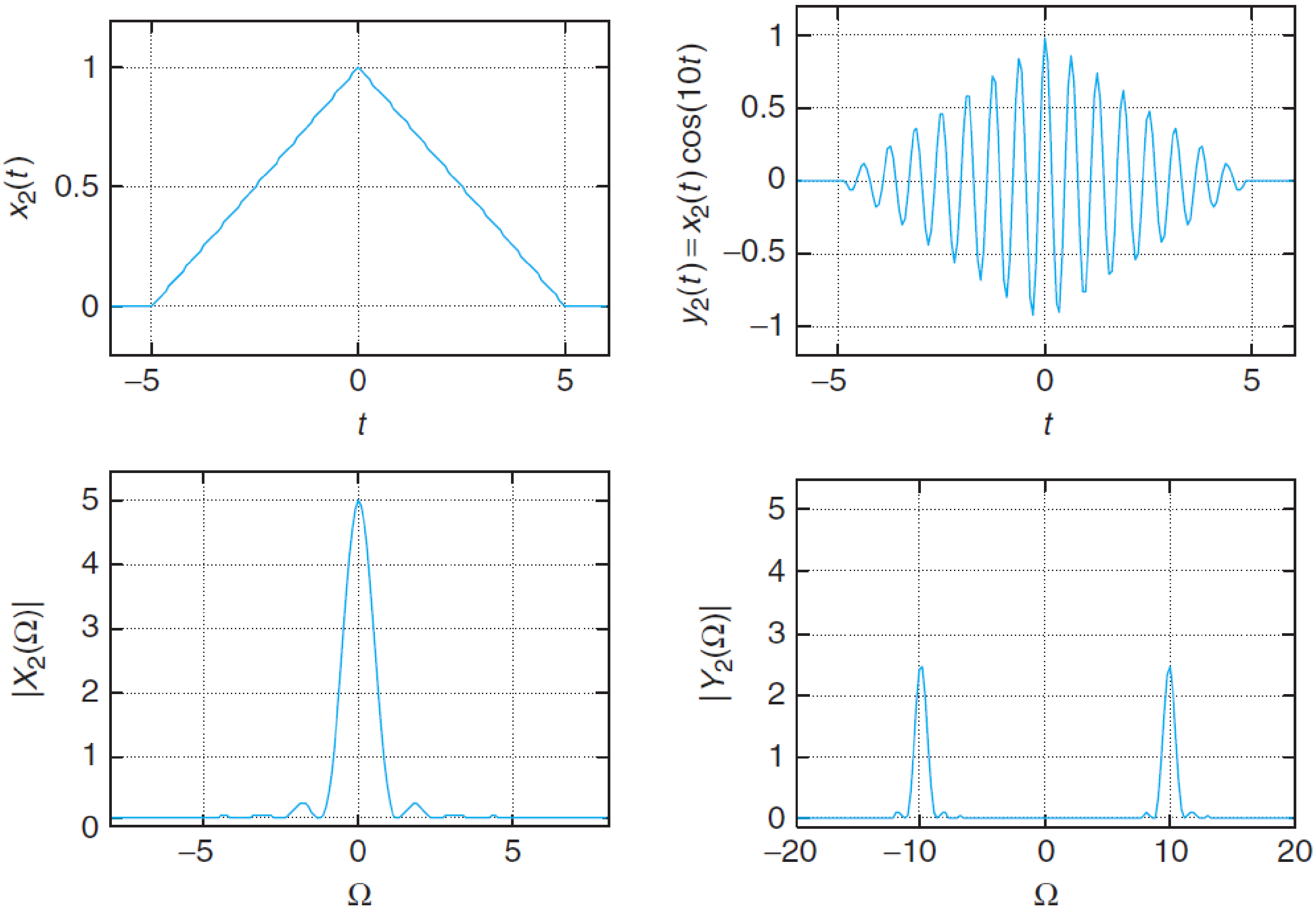
$$x(t)\cos(\Omega_0 t)$$

is given by

$$0.5\left[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)\right]$$

That is, $X(\Omega)$ is shifted to frequencies Ω_0 and $-\Omega_0$, and multiplied by 0.5.

Signal Modulation: Example



Fourier Transform of Periodic Signals

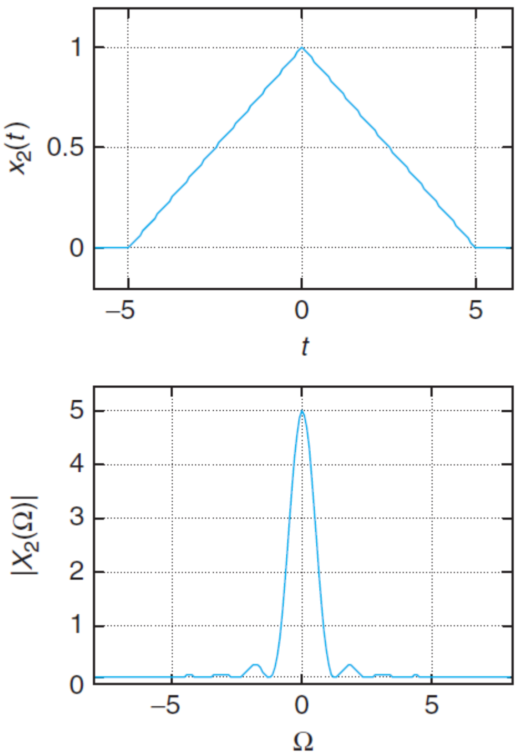
For a periodic signal $x(t)$ of period T_0, we have the Fourier pair

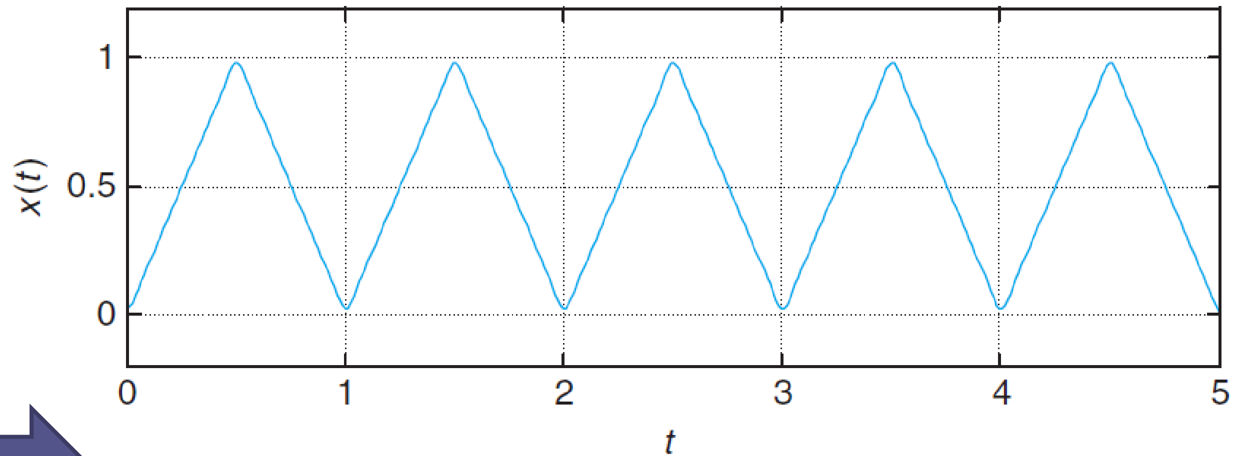
$$x(t) = \sum_k X_k e^{jk\Omega_0 t} \quad \Leftrightarrow \quad X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$$

obtained by representing $x(t)$ by its Fourier series.

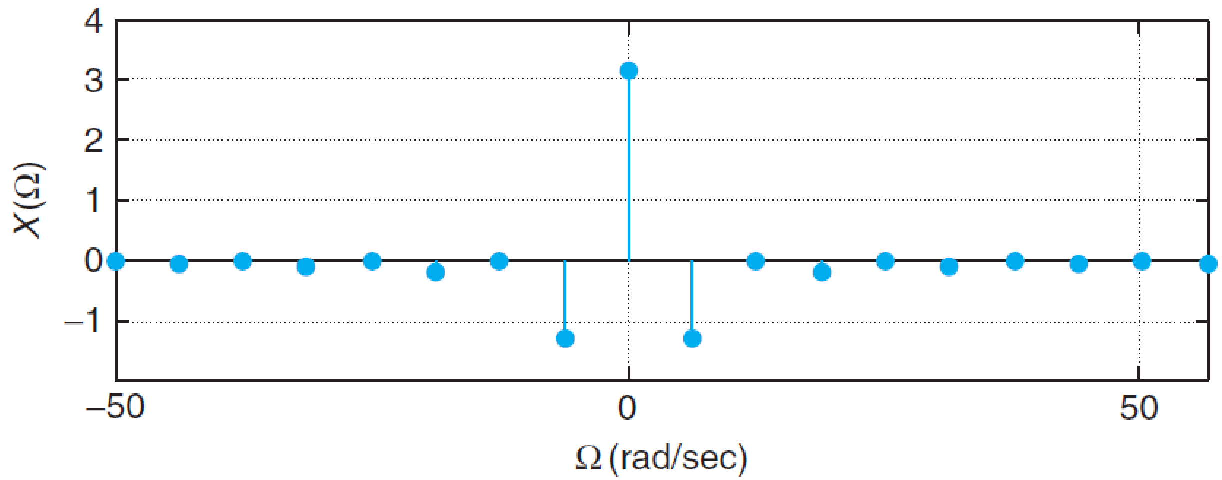
- Periodic Signals are represented by Sampled Fourier transform
- Sampled Signals are representing by Periodic Fourier Transform (from duality)

Fourier Transform of Periodic Signals: Example





(a)



Parseval's Energy Conservation

For a finite-energy signal $x(t)$ with Fourier transform $X(\Omega)$, its energy is conserved when going from the time to the frequency domain, or

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega \tag{5.15}$$

Thus, $|X(\Omega)|^2$ is an energy density indicating the amount of energy at each of the frequencies Ω. The plot $|X(\Omega)|^2$ versus Ω is called the energy spectrum of $x(t)$, and it displays how the energy of the signal is distributed over frequency.

- Energy in Time Domain = Energy in Frequency Domain

Symmetry of Spectral Representations

If $X(\Omega)$ is the Fourier transform of a real-valued signal $x(t)$, periodic or aperiodic, the magnitude $|X(\Omega)|$ is an even function of Ω:

$$|X(\Omega)| = |X(-\Omega)| \tag{5.16}$$

and the phase $\angle X(\Omega)$ is an odd function of Ω:

$$\angle X(\Omega) = -\angle X(-\Omega) \tag{5.17}$$

We then have:

Magnitude spectrum:	$\lvert X(\Omega)\rvert$ versus Ω
Phase spectrum:	$\angle X(\Omega)$ versus Ω
Energy/power spectrum:	$\lvert X(\Omega)\rvert^2$ versus Ω

- Clearly, if the signal is complex, the above symmetry will NOT hold

Convolution and Filtering

If the input $x(t)$ (periodic or aperiodic) to a stable LTI system has a Fourier transform $X(\Omega)$, and the system has a frequency response $H(j\Omega) = \mathcal{F}[h(t)]$ where $h(t)$ is the impulse response of the system, the output of the LTI system is the convolution integral $y(t) = (x * h)(t)$, with Fourier transform

$$Y(\Omega) = X(\Omega)\, H(j\Omega) \tag{5.18}$$

- Relation between transfer function and frequency response:

$$\begin{aligned} H(j\Omega) &= \mathcal{L}[h(t)]\big|_{s=j\Omega} \\ &= H(s)\big|_{s=j\Omega} \end{aligned} \quad \Leftrightarrow \quad H(j\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

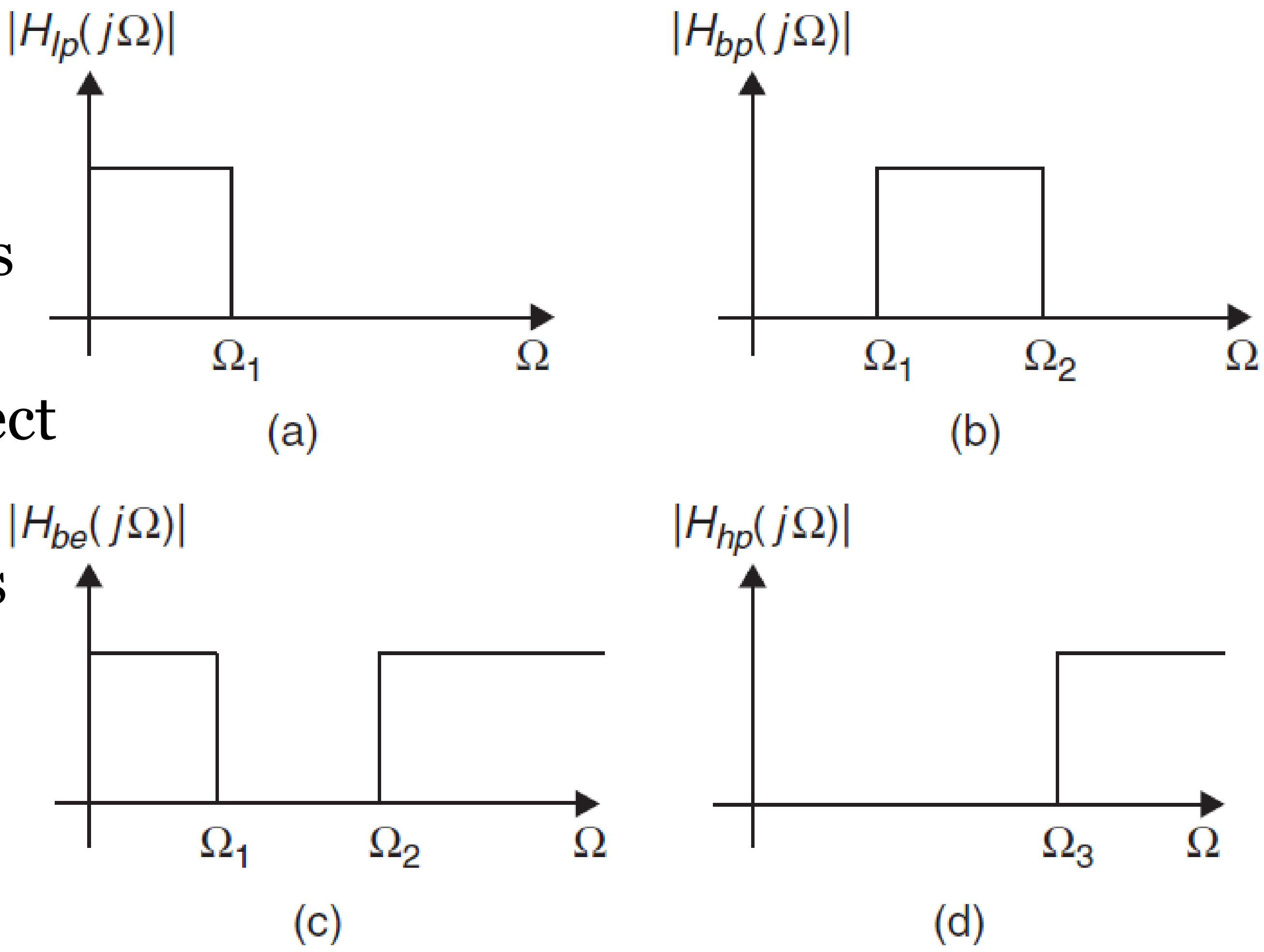
Basics of Filtering

- The filter design consists in finding a transfer function $H(s) = B(s)=A(s)$ that satisfies certain specifications that will allow getting rid of the noise. Such specifications are typically given in the frequency domain.

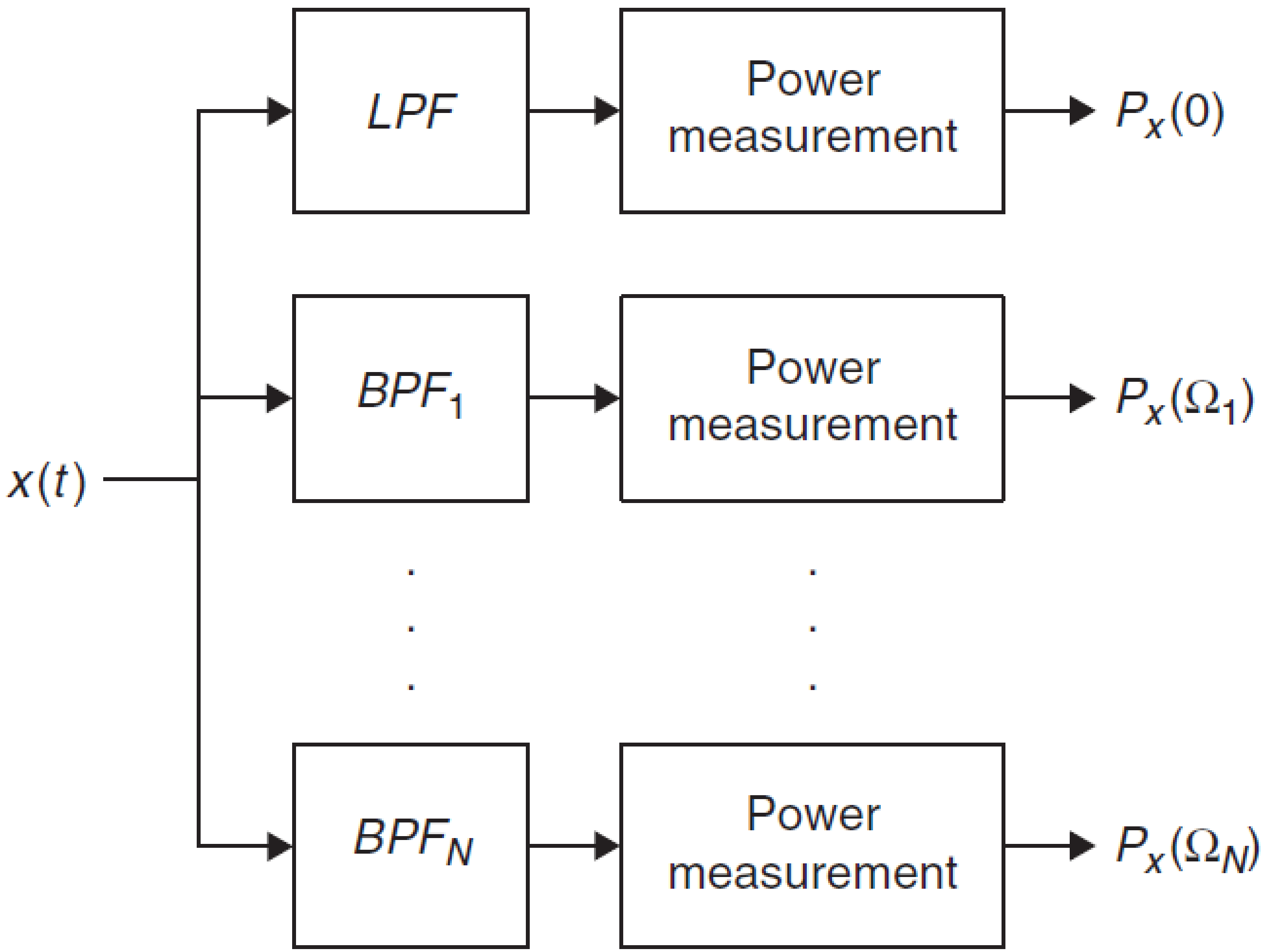
$$\boxed{Y(\Omega) = H(j\Omega)X(\Omega)}$$

Ideal Filters

- (a) Low-Pass
- (b) Band-Pass
- (c) Band-Reject
- (d) High-Pass



Spectrum Analyzer



Time Shifting Property

If $x(t)$ has a Fourier transform $X(\Omega)$, then

$$x(t - t_0) \Leftrightarrow X(\Omega)e^{-j\Omega t_0}$$

$$x(t + t_0) \Leftrightarrow X(\Omega)e^{j\Omega t_0}$$

- Example:

$$x(t) = A[\delta(t - \tau) + \delta(t + \tau)]$$

$$\Downarrow$$

$$X(\Omega) = A[1e^{-j\Omega\tau} + 1e^{j\Omega\tau}]$$

Differentiation and Integration

If $x(t)$, $-\infty < t < \infty$, has a Fourier tranform $X(\Omega)$, then

$$\frac{d^N x(t)}{dt^N} \Leftrightarrow (j\Omega)^N X(\Omega)$$

$$\int_{-\infty}^{t} x(\sigma)d\sigma \Leftrightarrow \frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$$

where

$$X(0) = \int_{-\infty}^{\infty} x(t)dt$$

Table 5.1 Basic Properties of the Fourier Transform

	Time Domain	Frequency Domain
Signals and constants	$x(t), y(t), z(t), \alpha, \beta$	$X(\Omega), Y(\Omega), Z(\Omega)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\Omega) + \beta Y(\Omega)$
Expansion/contraction in time	$x(\alpha t), \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} X\left(\frac{\Omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\Omega)$
Parseval's energy relation	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\Omega)\rvert^2 d\Omega$
Duality	$X(t)$	$2\pi x(-\Omega)$
Time differentiation	$\frac{d^n x(t)}{dt^n}$, $n \geq 1$, integer	$(j\Omega)^n X(\Omega)$
Frequency differentiation	$-jtx(t)$	$\frac{dX(\Omega)}{d\Omega}$
Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$
Time shifting	$x(t-\alpha)$	$e^{-j\alpha\Omega} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 t} x(t)$	$X(\Omega - \Omega_0)$
Modulation	$x(t)\cos(\Omega_c t)$	$0.5[X(\Omega - \Omega_c) + X(\Omega + \Omega_c)]$
Periodic signals	$x(t) = \sum_k X_k e^{jk\Omega_0 t}$	$X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$
Symmetry	$x(t)$ real	$\lvert X(\Omega)\rvert = \lvert X(-\Omega)\rvert$
		$\angle X(\Omega) = -\angle X(-\Omega)$
Convolution in time	$z(t) = [x * y](t)$	$Z(\Omega) = X(\Omega)Y(\Omega)$
Windowing/multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X * Y](\Omega)$
Cosine transform	$x(t)$ even	$X(\Omega) = \int_{-\infty}^{\infty} x(t)\cos(\Omega t)dt$, real
Sine transform	$x(t)$ odd	$X(\Omega) = -j\int_{-\infty}^{\infty} x(t)\sin(\Omega t)dt$, imaginary

Problem Assignments

- Problems: 5.4, 5.5, 5.6, 5.18, 5.20, 5.23
- Partial Solutions available from the student section of the textbook web site