A New Model-Based Algebraic Solution to the Gridding Problem

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Abstract We present a new approach for image reconstruction from non-uniformly sampled k-space. The image is assumed to be piecewise constant to model practical display using pixels. Using the continuous Fourier transform, the mapping of these unknown pixel values to the available frequency domain values is derived. Even though the system matrix of this problem can be shown to be dense and too large to solve for practical purposes, we observe that a simple orthogonal transformation to the rows of this matrix converts the matrix to a sparse matrix. This system is subsequently solved using the iterative conjugate gradient method.

Introduction

The most commonly used gridding technique is the one described in (1). Nevertheless, this technique relies on a rigid kernel that does not take into consideration the variations in sampling density between different areas in the k-space. Recently, several techniques have been proposed to address this problem (2,3) whereby a spatially variant kernel is used to perform the gridding. Even though these techniques showed a marked improvement over the classical technique, their solution cannot be claimed optimal for the problem at hand. In this work, we proposed a more realistic model for the problem and show its solution to be optimal. Moreover, we demonstrate the computational efficiency of the new method.

Theory

Consider *f(x,y)* as the continuous-space spatial domain intensity distribution and let the available k-space (frequency domain) samples be $F(kx_i,\ ky_i)$, where i=0,1,…,L-1, and L is the number of samples. Given the way the image is displayed, the spatial domain can be modeled as the sum of shifted gate-like functions representing the pixels of the image up to the desired resolution. That is,

$$f(x,y)=\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\alpha_{n,m}\cdot\Pi(x-x_n,y-y_n)$$

Consequently, the continuous Fourier transform of *f(x,y)* can be obtained as,

$$F(k_x,k_y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\alpha_{n,m}\Pi(x-x_n,y-y_m)\cdot e^{-j2\pi(k_x x+k_y y)}dxdy,$$

which reduces to,

$$F(k_x,k_y)=Sinc(ck_x)\cdot Sinc(dk_y)\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\alpha_{n,m}\cdot e^{-j2\pi(k_x x_n+k_y y_m)}.$$

Here *c* and *d* correspond to the pixel size. Hence, given the available nonuniform samples in the k-space, we can express the above equation as a linear system in the form $\mathbf{A}\vec{v}=\vec{b}$ whose equations are obtained by substituting the set of L value for k_x and k_y. Hence, the linear system matrix has dimensions of L rows and M·N columns. The $\vec{b}$ vector consists of the values of the acquired frequency samples indexed by their *i* value. Given this index and from the known locations of the samples in the k-space, the matrix **A** can be computed for a given arrangement of the pixel values in the *v* vector. The unknowns in this linear system are the pixel intensity values. In order to simplify the practical implementation of the solution, we observe that multiplication of the discrete Fourier transform matrix **H** whose inverse is simply its hermitian by the rows of the matrix result in compacting the energy within the row into

$$\vec{b}=\mathbf{A}\vec{v}=\mathbf{A}\cdot\mathbf{H}^H\cdot\mathbf{H}\cdot\vec{v}=(\mathbf{H}\cdot\mathbf{A}^H)^H\cdot\vec{V}=\mathbf{M}\cdot\vec{V},$$

only a small number of points instead of the original dense matrix form. To preserve the linear system unchanged while taking advantage of this property, let:

where **M** is sparse and the intermediate solution $\vec{V}$ is the 1-D discrete Fourier transform of the 1-D listing of pixel values.

Methods

The energy compacting property used above can be verified by evaluating one row of the linear system matrix and applying the Fourier transformation to it. We have observed that more than 90% of the energy is contained within less than 10 coefficients out of the much longer M·N row values. We allow the user to select the energy percentage and then perform the truncation automatically for each row. Notice that the size and location of the large coefficients that are selected to achieve the energy percentage varies from one row to another. This means that the kernel used to perform the mapping is spatially varying. The non-zero coefficients are stored efficiently using sparse matrix techniques as a table containing their locations and values. The results show that the number of operations required to perform any matrix computation using this format is ***O***(L).

The conjugate gradient technique is used to solve this matrix (4). Assuming the initial solution to be zero, the first iteration corresponds to simply premultiplying the vector *b* by the hermitian of the matrix **M.** This first step usually provides a reasonable approximate solution. In case a number of iterations *r* is used, the complexity of the algorithm remains ***O***(*r*L), which remains close to conventional techniques for small values of *r*.

Experimental Results

The proposed technique was implemented to reconstruct images of a numerical phantom as well as actual data acquired using a spiral imaging sequence. The preliminary results support the theory and show a large potential for its clinical use.

References

1. Jackson JI et al, *IEEE Trans. Med. Imag.* 10(3): 473-478, 1991.
2. Rosenfeld D, *Magn. Reson. Med.* 40:14-23, 1998.
3. Sedarat H and Nishimura DG, *IEEE Trans. Med. Imag.* 19(4): 306-317, 2000.
4. Kadah YM, Hu X, *IEEE Trans. Med Imag.* 17(3): 362-370, 1998.

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