Fourier Transform Definition

$x(t) \leftrightarrow X(\Omega)$

where the signal $x(t)$ is transformed into a function $X(\Omega)$ in the frequency domain by the

Fourier transform: $X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} \, dt$

while $X(\Omega)$ is transformed into a signal $x(t)$ in the time domain by the

Inverse Fourier transform: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t} \, d\Omega$
Fourier Transforms from Laplace Transforms

- If the region of convergence (ROC) of the Laplace transform $X(s)$ contains the $j\Omega$ axis, so that $X(s)$ can be defined for $s = D j\Omega$, then:

\[
\mathcal{F}[x(t)] = \mathcal{L}[x(t)] \bigg|_{s=j\Omega} = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} \, dt
\]

\[
= X(s) \bigg|_{s=j\Omega}
\]
<table>
<thead>
<tr>
<th>Function of Time</th>
<th>Function of $\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 $\delta(t)$</td>
<td>1</td>
</tr>
<tr>
<td>2 $\delta(t - \tau)$</td>
<td>$e^{-j\Omega \tau}$</td>
</tr>
<tr>
<td>3 $u(t)$</td>
<td>$\frac{1}{j\Omega} + \pi \delta(\Omega)$</td>
</tr>
<tr>
<td>4 $u(-t)$</td>
<td>$\frac{-1}{j\Omega} + \pi \delta(\Omega)$</td>
</tr>
<tr>
<td>5 $\text{sgn}(t) = 2[u(t) - 0.5]$</td>
<td>$\frac{2}{j\Omega}$</td>
</tr>
<tr>
<td>6 $A$, $-\infty &lt; t &lt; \infty$</td>
<td>$2\pi A \delta(\Omega)$</td>
</tr>
<tr>
<td>7 $Ae^{-at}u(t)$, $a &gt; 0$</td>
<td>$\frac{A}{j\Omega + a}$</td>
</tr>
<tr>
<td>8 $Ate^{-at}u(t)$, $a &gt; 0$</td>
<td>$\frac{A}{(j\Omega + a)^2}$</td>
</tr>
<tr>
<td>9 $e^{-a</td>
<td>t</td>
</tr>
<tr>
<td>10 $\cos(\Omega_0 t)$, $-\infty &lt; t &lt; \infty$</td>
<td>$\pi \left[ \delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0) \right]$</td>
</tr>
<tr>
<td>11 $\sin(\Omega_0 t)$, $-\infty &lt; t &lt; \infty$</td>
<td>$-j\pi \left[ \delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0) \right]$</td>
</tr>
<tr>
<td>12 $A[u(t + \tau) - u(t - \tau)]$, $\tau &gt; 0$</td>
<td>$2A\tau \frac{\sin(\Omega \tau)}{\Omega \tau}$</td>
</tr>
<tr>
<td>13 $\frac{\sin(\Omega_0 t)}{\pi t}$</td>
<td>$u(\Omega + \Omega_0) - u(\Omega - \Omega_0)$</td>
</tr>
<tr>
<td>14 $x(t) \cos(\Omega_0 t)$</td>
<td>$0.5 \left[ X(\Omega - \Omega_0) + X(\Omega + \Omega_0) \right]$</td>
</tr>
</tbody>
</table>
Linearity

- Fourier transform is a linear operator
- Superposition holds

If $\mathcal{F}[x(t)] = X(\Omega)$ and $\mathcal{F}[y(t)] = Y(\Omega)$, for constants $\alpha$ and $\beta$, we have that

$$\mathcal{F}[\alpha x(t) + \beta y(t)] = \alpha \mathcal{F}[x(t)] + \beta \mathcal{F}[y(t)]$$

$$= \alpha X(\Omega) + \beta Y(\Omega)$$
Inverse Proportionality of Time and Frequency

• Support of $X(\Omega)$ is inversely proportional to support of $x(t)$
• If $x(t)$ has a Fourier transform $X(\Omega)$ and $\alpha \neq 0$ is a real number, then $x(\alpha t)$ is:
  ■ Contracted $(\alpha > 1)$,
  ■ Contracted and reflected $(\alpha < -1)$,
  ■ Expanded $(0 < \alpha < 1)$,
  ■ Expanded and reflected $(-1 < \alpha < 0)$, or
  ■ Simply reflected $(\alpha = -1)$

• Then,

$$x(\alpha t) \iff \frac{1}{|\alpha|} X\left(\frac{\Omega}{\alpha}\right)$$
Inverse Proportionality of Time and Frequency - Example

- Fourier transform of 2 pulses of different width
  - 4-times wider pulse have 4-times narrower Fourier transform
Duality

- By interchanging the frequency and the time variables in the definitions of the direct and the inverse Fourier transform similar equations are obtained.
- Thus, the direct and the inverse Fourier transforms are dual.

\[ x(t) \Leftrightarrow X(\Omega) \]

\[ X(t) \Leftrightarrow 2\pi x(-\Omega) \]
Duality: Example

![Graphs showing duality example]
Signal Modulation

- Frequency shift: If $X(\Omega)$ is the Fourier transform of $x(t)$, then we have the pair
  $$x(t)e^{j\Omega_0 t} \Leftrightarrow X(\Omega \pm \Omega_0)$$

- Modulation: The Fourier transform of the modulated signal
  $$x(t) \cos(\Omega_0 t)$$

is given by

$$0.5 \left[ X(\Omega - \Omega_0) + X(\Omega + \Omega_0) \right]$$

That is, $X(\Omega)$ is shifted to frequencies $\Omega_0$ and $-\Omega_0$, and multiplied by 0.5.
Signal Modulation: Example

- $x_2(t) = \frac{\sin(\Omega t)}{\Omega}$
- $y_2(t) = x_2(t) \cos(10t)$

- $|X_2(\Omega)|$ and $|Y_2(\Omega)|$ as functions of frequency $\Omega$. 
Fourier Transform of Periodic Signals

For a periodic signal $x(t)$ of period $T_0$, we have the Fourier pair

$$x(t) = \sum_k X_k e^{jk\Omega_0 t} \quad \Leftrightarrow \quad X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$$

obtained by representing $x(t)$ by its Fourier series.

- Periodic Signals are represented by Sampled Fourier transform
- Sampled Signals are representing by Periodic Fourier Transform (from duality)
Fourier Transform of Periodic Signals: Example
Parseval’s Energy Conservation

For a finite-energy signal $x(t)$ with Fourier transform $X(\Omega)$, its energy is conserved when going from the time to the frequency domain, or

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega$$

(5.15)

Thus, $|X(\Omega)|^2$ is an energy density indicating the amount of energy at each of the frequencies $\Omega$. The plot $|X(\Omega)|^2$ versus $\Omega$ is called the energy spectrum of $x(t)$, and it displays how the energy of the signal is distributed over frequency.

- Energy in Time Domain = Energy in Frequency Domain
Symmetry of Spectral Representations

If $X(\Omega)$ is the Fourier transform of a real-valued signal $x(t)$, periodic or aperiodic, the magnitude $|X(\Omega)|$ is an even function of $\Omega$:

$$|X(\Omega)| = |X(-\Omega)|$$  \hspace{1cm} (5.16)

and the phase $\angle X(\Omega)$ is an odd function of $\Omega$:

$$\angle X(\Omega) = -\angle X(-\Omega)$$  \hspace{1cm} (5.17)

We then have:

- Magnitude spectrum: $|X(\Omega)|$ versus $\Omega$
- Phase spectrum: $\angle X(\Omega)$ versus $\Omega$
- Energy/power spectrum: $|X(\Omega)|^2$ versus $\Omega$

• Clearly, if the signal is complex, the above symmetry will NOT hold.
Convolution and Filtering

If the input \( x(t) \) (periodic or aperiodic) to a stable LTI system has a Fourier transform \( X(\Omega) \), and the system has a frequency response \( H(j\Omega) = \mathcal{F}[h(t)] \) where \( h(t) \) is the impulse response of the system, the output of the LTI system is the convolution integral \( y(t) = (x * h)(t) \), with Fourier transform

\[
Y(\Omega) = X(\Omega) H(j\Omega)
\]  

(5.18)

- Relation between transfer function and frequency response:

\[
H(j\Omega) = \mathcal{L}[h(t)]|_{s=j\Omega} = H(s)|_{s=j\Omega}
\]

\[
H(j\Omega) = \frac{Y(\Omega)}{X(\Omega)}
\]
Basics of Filtering

• The filter design consists in finding a transfer function $H(s) = B(s) = A(s)$ that satisfies certain specifications that will allow getting rid of the noise. Such specifications are typically given in the frequency domain.

\[ Y(\Omega) = H(j\Omega)X(\Omega) \]
Ideal Filters

- (a) Low-Pass
- (b) Band-Pass
- (c) Band-Reject
- (d) High-Pass
Spectrum Analyzer

\[ x(t) \rightarrow LPF \rightarrow Power \ measurement \rightarrow P_x(0) \]

\[ x(t) \rightarrow BPF_1 \rightarrow Power \ measurement \rightarrow P_x(\Omega_1) \]

\[ \vdots \]

\[ x(t) \rightarrow BPF_N \rightarrow Power \ measurement \rightarrow P_x(\Omega_N) \]
Time Shifting Property

If $x(t)$ has a Fourier transform $X(\Omega)$, then

\[
x(t - t_0) \Leftrightarrow X(\Omega)e^{-j\Omega t_0} \\
x(t + t_0) \Leftrightarrow X(\Omega)e^{j\Omega t_0}
\]

• Example:

\[
x(t) = A[\delta(t - \tau) + \delta(t + \tau)]
\]

\[
X(\Omega) = A[1e^{-j\Omega \tau} + 1e^{j\Omega \tau}]
\]
Differentiation and Integration

If $x(t)$, $-\infty < t < \infty$, has a Fourier transform $X(\Omega)$, then

$$
\frac{d^N x(t)}{dt^N} \quad \Leftrightarrow \quad (j\Omega)^N X(\Omega)
$$

$$
\int_{-\infty}^{t} x(\sigma) d\sigma \quad \Leftrightarrow \quad \frac{X(\Omega)}{j\Omega} + \pi X(0) \delta(\Omega)
$$

where

$$
X(0) = \int_{-\infty}^{\infty} x(t) dt
$$
<table>
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<td>Sine transform</td>
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Problem Assignments

- Problems: 5.4, 5.5, 5.6, 5.18, 5.20, 5.23
- Partial Solutions available from the student section of the textbook web site