Digital Signal Processing - Chapter 9

The Z-Transform

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Z-Transform

• Just as with the Laplace transform for continuous-time signals and systems, the Z-transform provides a way to represent discrete-time signals and systems, and to process discrete-time signals
  ▫ Although the Z-transform can be related to the Laplace transform, the relation is operationally not very useful
• Representation of discrete-time signals by Z-transform is very intuitive—it converts a sequence of samples into a polynomial
  ▫ As with Laplace transform and convolution integral, the most important property of the Z-transform is the implementation of the convolution sum as a multiplication of polynomials
Laplace Transform of Sampled Signals

• Consider a sampled signal:

\[ x(t) = \sum_{n} x(n T_s) \delta(t - n T_s) \]

• Then,

\[ X(s) = \sum_{n} x(n T_s) \mathcal{L}[\delta(t - n T_s)] = \sum_{n} x(n T_s) e^{-n s T_s} \]

• Let \( z = e^{s T_s} \), then this is called the Z-transform of \( x(n) \):

\[ \mathcal{Z}[x(n T_s)] = \mathcal{L}[x_s(t)] \bigg|_{z = e^{s T_s}} = \sum_{n} x(n T_s) z^{-n} \]
Comments About Z-Transform

- Letting \( s = j\Omega \), we find that the Fourier transform is a special case when \( z = e^{j\Omega} \)
  - Periodic Fourier transform since \( x(t) \) is sampled
- While Laplace transform may have an infinite number of poles or zeros—complicating the partial fraction expansion when finding its inverse, the inverse Z-transform can be readily obtained using the time-shift property from the \( z \) polynomial:

\[
\mathcal{Z}[x(nT_s)] = \mathcal{L}[x_s(t)]\bigg|_{z = e^{sT_s}} = \sum_n x(nT_s)z^{-n}
\]

\[
x(t) = \sum_n x(nT_s)\delta(t - nT_s)
\]
z-Plane vs. s-Plane

- Connection between the s-plane and the z-plane

\[ z = e^{sT_s} = e^{(\sigma + j\Omega)T_s} = e^{\sigma T_s} e^{j\Omega T_s} \]
Forward Z-Transform Definitions

• Two-sided

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

defined in a region of convergence (ROC) in the z-plane.

• One-Sided

\[ X_1(z) = \mathcal{Z}(x[n]u[n]) = \sum_{n=0}^{\infty} x[n]u[n]z^{-n} \]

defined in a region of convergence (ROC) in the z-plane.
Region of Convergence

• The infinite summation of the two-sided Z-transform must converge for some values of z
  ▫ For \( X(z) \) to converge it is necessary that:

\[
|X(z)| = \left| \sum_n x[n] z^{-n} \right| \leq \sum_n |x[n]| |r^{-n} e^{j\omega n}| = \sum_n |x[n]| |r^{-n}| < \infty
\]

• Poles and zeros

The poles of a Z-transform \( X(z) \) are complex values \( \{p_k\} \) such that

\[
X(p_k) \to \infty
\]

while the zeros of \( X(z) \) are the complex values \( \{z_k\} \) that make

\[
X(z_k) = 0
\]
Poles and Zeros: Example

- Find the poles and zeros of the following Z-transforms:

\[(a) \quad X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} \]

\[(b) \quad X_2(z) = \frac{(z^{-1} - 1)(z^{-1} + 2)^2}{z^{-1}(z^{-2} + \sqrt{2}z^{-1} + 1)} \]

\[X_1(z) = \frac{z^3(1 + 2z^{-1} + 3z^{-2} + 4z^{-3})}{z^3} = \frac{z^3 + 2z^2 + 3z + 4}{z^3} = \frac{N_1(z)}{D_1(z)} \]

three poles at \(z = 0\)

zeros are the roots of \(N_1(z)\)

\[X_2(z) = \frac{z^3(z^{-1} - 1)(z^{-1} + 2)^2}{z^3(z^{-1}(z^{-2} + \sqrt{2}z^{-1} + 1))} = \frac{(1 - z)(1 + 2z)^2}{1 + \sqrt{2}z + z^2} = \frac{N_2(z)}{D_2(z)} \]

poles of \(X_2(z)\) are the roots of \(D_2(z) = 1 + \sqrt{2}z + z^2 = 0\)

zeros of \(X_2(z)\) are the roots of \(N_2(z) = (1 - z)(1 + 2z)^2 = 0\)
ROC of Finite-Support Signals

- The ROC of the Z-transform of a signal $x[n]$ of finite support $[N_0, N_1]$ where $-\infty < N_0 < n < N_1 < \infty$,

$$X(z) = \sum_{n=N_0}^{N_1} x[n]z^{-n}$$

is the whole z-plane, excluding the origin $z= 0$ and/or $z = \pm \infty$ depending on $N_0$ and $N_1$
- Example:

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 9 \\ 0 & \text{otherwise} \end{cases} \quad X(z) = \sum_{n=0}^{9} 1 \ z^{-n}$$

ROC: Whole z plane except origin
ROC of Infinite-Support Signals

- Signals of infinite support are either causal, anti-causal, or a combination of these or non-causal
- Z-transform of a causal signal $x_c[n]$: 
  \[ X_c(z) = \sum_{n=0}^{\infty} x_c[n]z^{-n} = \sum_{n=0}^{\infty} x_c[n]r^{-n}e^{-jn\omega} \]
- Let $R_1$ be the radius of the farthest-out pole of $X_c(z)$,
  \[ |X_c(z)| \leq \sum_{n=0}^{\infty} |x_c[n]| |r^{-n}| < M \sum_{n=0}^{\infty} \left| \frac{R_1}{r} \right|^n < \infty \]
  \[ R_1/r < 1 \quad \Rightarrow \quad |z| = r > R_1 \]
- Anti-causal $x_a[n]$: ROC is the opposite: \[ |z| = r < R_2 \]
ROC of Infinite-Support Signals

- If the signal $x[n]$ is non-causal, it can be expressed as,
  \[ x[n] = x_c[n] + x_a[n] \]

- ROC: combination of causal and anti-causal ROCs,
  \[ 0 \leq R_1 < |z| < R_2 < \infty \]

For the Z-transform $X(z)$ of an infinite-support signal:

- A causal signal $x[n]$ has a region of convergence $|z| > R_1$ where $R_1$ is the largest radius of the poles of $X(z)$—that is, the region of convergence is the outside of a circle of radius $R_1$.
- An anti-causal signal $x[n]$ has as region of convergence the inside of the circle defined by the smallest radius $R_2$ of the poles of $X(z)$, or $|z| < R_2$.
- A noncausal signal $x[n]$ has as region of convergence $R_1 < |z| < R_2$, or the inside of a torus of inside radius $R_1$ and outside radius $R_2$ corresponding to the maximum and minimum radii of the poles of $X_c(z)$ and $X_a(z)$, which are the Z-transforms of the causal and anti-causal components of $x[n]$. 
ROC: Example

- Find ROC of the Z-transforms of the following signals:

(a) \( x_1[n] = \left( \frac{1}{2} \right)^n u[n] \)

\( x_1[n] \) is causal

\[
X_1(z) = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n z^{-n} = \frac{1}{1 - 0.5z^{-1}} = \frac{z}{z - 0.5}
\]

\( \mathcal{R}_1 : |z| > 0.5 \)

(b) \( x_2[n] = -\left( \frac{1}{2} \right)^n u[-n - 1] \)

\( x_2[n] \) is anti-causal.

\[
X_2(z) = -\sum_{n=-\infty}^{-1} \left( \frac{1}{2} \right)^n z^{-n} = -\sum_{m=0}^{\infty} \left( \frac{1}{2} \right)^{-m} z^m + 1
\]

\[
= -\sum_{m=0}^{\infty} 2^m z^m + 1 = \frac{-1}{1 - 2z} + 1 = \frac{z}{z - 0.5}
\]

\( \mathcal{R}_2 : |z| < 0.5 \)

*Note: ROC for \( x_1[n] + x_2[n] \) is empty: Z-transform does not exist for this sum!!*
The Z-transform is a linear transformation, meaning that

\[ \mathcal{Z}(ax[n] + by[n]) = a\mathcal{Z}(x[n]) + b\mathcal{Z}(y[n]) \]

for signals \( x[n] \) and \( y[n] \) and constants \( a \) and \( b \).

- Convolution: similar to Laplace and Fourier transforms

\[
  y[n] = [x * h][n] = \sum_{k=0}^{n} x[k]h[n - k] = \sum_{k=0}^{n} h[k]x[n - k]
\]

\[
  Y(z) = \mathcal{Z}\{[x * h][n]\} = \mathcal{Z}\{x[n]\}\mathcal{Z}\{h[n]\} = X(z)H(z)
\]

\[
  H(z) = \frac{Y(z)}{X(z)} = \frac{\mathcal{Z}\{output \ y[n]\}}{\mathcal{Z}\{input \ x[n]\}}
\]
Convolution Sum As a Polynomial Multiplication

- Consider \( X_1(z) = 1 + a_1 z^{-1} + a_2 z^{-2} \) and \( X_2(z) = 1 + b_1 z^{-1} \)

\[
X_1(z)X_2(z) = 1 + b_1 z^{-1} + a_1 z^{-1} + a_1 b_1 z^{-2} + a_2 z^{-2} + a_2 b_1 z^{-3}
\]

\[
= 1 + (b_1 + a_1) z^{-1} + (a_1 b_1 + a_2) z^{-2} + a_2 b_1 z^{-3}
\]

- The convolution sum of the two sequences \([1 \ a_1 \ a_2]\) and \([1 \ b_1]\), formed by the coefficients of \(X_1(z)\) and \(X_2(z)\), is given as \([1 \ (a_1+b_1) \ (a_2+b_1 \ a_1) \ a_2]\), which corresponds to the coefficients of the product of the polynomials \(X_1(z)X_2(z)\)

- Notice that the sequence of length 3 and the sequence of length 2 when convolved give a sequence of length 3+2-1=4
Finite Impulse Response (FIR) Filter

- A finite-impulse response or FIR filter is implemented by means of the convolution sum
- Consider an FIR with an input–output equation:

\[ y[n] = \sum_{k=0}^{N-1} b_k x[n - k] \]

- Impulse response: let \( x[n] = \delta[n] \)

\[ h[n] = \sum_{k=0}^{N-1} b_k \delta[n - k] \]

- Hence,

\[ y[n] = \sum_{k=0}^{N-1} h[k] x[n - k] \]

\[ Y(z) = H(z)X(z) \]
Convolution Sum Length

- The length of the convolution sum of two sequences of lengths M and N is M+N-1
- If one of the sequences is of infinite length, the length of the convolution is infinite
- Thus, for an *Infinite Impulse Response (IIR)* or recursive filters the output is always of infinite length for any input signal, given that the impulse response of these filters is of infinite length
Interconnecting Discrete-Time Systems

(a) \[ x[n] \xrightarrow{H_1(z)} H_2(z) \xrightarrow{y[n]} = x[n] \xrightarrow{H_2(z)} H_1(z) \xrightarrow{y[n]} = x[n] \xrightarrow{H_1(z)H_2(z)} y[n] \]

(b) \[ x[n] \xrightarrow{H_1(z)} H_2(z) \xrightarrow{y[n]} + y[n] = x[n] \xrightarrow{H_1(z)+H_2(z)} y[n] \]

(c) \[ x[n] \xrightarrow{H_1(z)} H_2(z) \xrightarrow{y[n]} \]

\[ x[n] \xrightarrow{w[n]} e[n] \]

\[ \frac{H_1(z)}{1 + H_1(z)H_2(z)} \xrightarrow{y[n]} = x[n] \]
Initial and Final Value Properties

Initial value: \( x[0] = \lim_{z \to \infty} X(z) \)

Final value: \( \lim_{n \to \infty} x[n] = \lim_{z \to 1} (z - 1)X(z) \)

\[
\lim_{z \to \infty} X(z) = \lim_{z \to \infty} \left( x[0] + \sum_{n=1}^{\infty} \frac{x[n]}{z^n} \right) = x[0]
\]

\[
(z - 1)X(z) = \sum_{n=0}^{\infty} x[n]z^{-n+1} - \sum_{n=0}^{\infty} x[n]z^{-n}
\]

\[
= x[0]z + \sum_{n=0}^{\infty} [x[n+1] - x[n]]z^{-n}
\]

\[
\lim_{z \to 1} (z - 1)X(z) = x[0] + \sum_{n=0}^{\infty} (x[n+1] - x[n])
\]

\[
= x[0] + (x[1] - x[0]) + (x[2] - x[1]) + (x[3] - x[2]) + \cdots
\]

\[
= \lim_{n \to \infty} x[n]
\]
Table 9.1 One-Sided Z-Transforms

<table>
<thead>
<tr>
<th>Function of Time</th>
<th>Function of $z$, ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\delta[n]$</td>
<td>1, whole $z$-plane</td>
</tr>
<tr>
<td>2. $u[n]$</td>
<td>$\frac{1}{1-z^{-1}}$, $</td>
</tr>
<tr>
<td>3. $nu[n]$</td>
<td>$\frac{z^{-1}}{(1-z^{-1})^2}$, $</td>
</tr>
<tr>
<td>4. $n^2u[n]$</td>
<td>$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$, $</td>
</tr>
<tr>
<td>5. $\alpha^n u[n]$, $</td>
<td>\alpha</td>
</tr>
<tr>
<td>6. $n\alpha^n u[n]$, $</td>
<td>\alpha</td>
</tr>
<tr>
<td>7. $\cos(\omega_0 n)u[n]$</td>
<td>$\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$, $</td>
</tr>
<tr>
<td>8. $\sin(\omega_0 n)u[n]$</td>
<td>$\frac{\sin(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$, $</td>
</tr>
<tr>
<td>9. $\alpha^n \cos(\omega_0 n)u[n]$, $</td>
<td>\alpha</td>
</tr>
<tr>
<td>10. $\alpha^n \sin(\omega_0 n)u[n]$, $</td>
<td>\alpha</td>
</tr>
<tr>
<td>Property</td>
<td>Equation</td>
</tr>
<tr>
<td>--------------------------------</td>
<td>---------------------------------------------------------</td>
</tr>
<tr>
<td>Causal signals and constants</td>
<td>$\alpha x[n], \beta y[n]$</td>
</tr>
<tr>
<td>Linearity</td>
<td>$\alpha x[n] + \beta y[n]$</td>
</tr>
<tr>
<td>Convolution sum</td>
<td>$(x * y)[n] = \sum_k x[n] y[n - k]$</td>
</tr>
<tr>
<td>Time shifting—causal</td>
<td>$x[n - N] \quad N \text{ integer}$</td>
</tr>
<tr>
<td>Time shifting—noncausal</td>
<td>$x[n] \quad \text{noncausal, } N \text{ integer}$</td>
</tr>
<tr>
<td>Time reversal</td>
<td>$x[-n]$</td>
</tr>
<tr>
<td>Multiplication by $n$</td>
<td>$n \cdot x[n]$</td>
</tr>
<tr>
<td>Multiplication by $n^2$</td>
<td>$n^2 \cdot x[n]$</td>
</tr>
<tr>
<td>Finite difference</td>
<td>$x[n] - x[n - 1]$</td>
</tr>
<tr>
<td>Accumulation</td>
<td>$\sum_{k=0}^{n} x[k]$</td>
</tr>
<tr>
<td>Initial value</td>
<td>$x[0]$</td>
</tr>
<tr>
<td>Final value</td>
<td>$\lim_{n \to \infty} x[n]$</td>
</tr>
<tr>
<td></td>
<td>$\lim_{z \to 1} (z - 1)X(z)$</td>
</tr>
</tbody>
</table>
Inverse Z-Transform (One-Sided)

- **Method #1**: If the Z-transform is given as a finite-order polynomial, the inverse can be found by inspection

\[
X(z) = \sum_{n=0}^{N} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots + x[N]z^{-N}
\]

- **Example:**

\[
X(z) = 1 + 2z^{-10} + 3z^{-20} \quad \Rightarrow \quad x[n] = \delta[n] + 2\delta[n - 10] + 3\delta[n - 20]
\]
Inverse Z-Transform (One-Sided)

- **Method #2**: Partial Fraction Expansion for rational functions given as \( X(z) = B(z)/A(z) \)
- To find the inverse we simply divide the polynomial \( B(z) \) by \( (z) \) to obtain a possible infinite-order polynomial in negative powers of \( z^{-1} \)
  - Coefficients of this polynomial are the inverse values
- Disadvantage: it does not provide a closed-form solution
  - Useful when interested to get a few initial values of \( x[n] \)
- Example:

\[
X(z) = \frac{1}{1 + 2z^{-2}} = 1 + (-2z^{-2})^1 + (-2z^{-2})^2 + (-2z^{-2})^3 + \cdots
\]

\[
x[0] = 1 \\
x[1] = 0 \\
x[2] = -2 \\
x[3] = 0 \\
x[4] = (-2)^2
\]
Inverse Z-Transform (One-Sided)

• Method #3: Partial Fraction Expansion
  ▫ Similar to Laplace transform

\[ X(z) = \frac{N(z)}{D(z)} \]

• Example:

\[ X(z) = \frac{1 + z^{-1}}{(1 + 0.5z^{-1})(1 - 0.5z^{-1})} \]

\[ X(z) = \frac{1 + z^{-1}}{(1 + 0.5z^{-1})(1 - 0.5z^{-1})} = \frac{A}{1 + 0.5z^{-1}} + \frac{B}{1 - 0.5z^{-1}} \]

\[ x[n] = [-0.5(-0.5)^n + 1.5(0.5)^n]u[n] \]
Solution of Difference Equations

- Use the shifting in time property of the Z-transform in the solution of difference equations with initial conditions
  - Very similar to Laplace transform when solving differential equations

<table>
<thead>
<tr>
<th>Time shifting—causal</th>
<th>$x[n - N]$ $N$ integer</th>
<th>$z^{-N}X(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time shifting—noncausal</td>
<td>$x[n - N]$</td>
<td>$z^{-N}X(z) + x[-1]z^{-N+1}$</td>
</tr>
<tr>
<td></td>
<td>$x[n]$ noncausal, $N$ integer</td>
<td>$+ x[-2]z^{-N+2} + \ldots + x[-N]$</td>
</tr>
</tbody>
</table>

$$y[n] = y_{zs}[n] + y_{zi}[n]$$
Solution of Difference Equations: Example 1

• Solve the following difference equation with zero initial conditions and $x[n]=u[n]$

$$y[n] = y[n - 1] - 0.25y[n - 2] + x[n] \quad n \geq 0$$

Solution:

$$Y(z) = \frac{X(z)}{1 - z^{-1} + 0.25z^{-2}}$$

$$= \frac{1}{(1 - z^{-1})(1 - z^{-1} + 0.25z^{-2})} = \frac{z^3}{(z - 1)(z^2 - z + 0.25)} \quad |z| > 1$$

$$y[n] = Au[n] + [B(0.5)^n + Cn(0.5)^n] u[n]$$
Solution of Difference Equations: Example 2

• Fine the complete response for the following difference equation:
  \[ y[n] + y[n - 1] - 4y[n - 2] - 4y[n - 3] = 3x[n] \quad n \geq 0 \]
  \[ y[-1] = 1 \]
  \[ y[-2] = y[-3] = 0 \]

• Solution:
  \[ x[n] = u[n] \]

  \[ Y(z)[1 + z^{-1} - 4z^{-2} - 4z^{-3}] = 3X(z) + [-1 + 4z^{-1} + 4z^{-2}] \]

  \[ Y(z) = 3 \frac{X(z)}{A(z)} + \frac{-1 + 4z^{-1} + 4z^{-2}}{A(z)} \quad |z| > 2 \]

  \[ A(z) = 1 + z^{-1} - 4z^{-2} - 4z^{-3} = (1 + z^{-1})(1 + 2z^{-1})(1 - 2z^{-1}) \]

  \[ y[n] = y_{zs}[n] + y_{zi}[n] \]
Problem Assignments

- Partial Solutions available from the student section of the textbook web site