Signals and Systems - Chapter 1

Continuous-Time Signals

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Overview of Chapter 1

• Mathematical representation of signals
• Classification of signals
• Signal manipulation
• Basic signal representation
Introduction

• Learning how to represent signals in analog as well as in digital forms and how to model and design systems capable of dealing with different types of signals
• Most signals come in analog form
• Trend has been toward digital representation and processing of data
  ▫ Computer capabilities increase continuously
Analog vs. Discrete Signals

• **Analog:** Infinitesimal calculus (or just calculus)
  ▫ Functions of continuous variables
  ▫ Derivative
  ▫ Integral
  ▫ Differential equations

• **Discrete:** Finite calculus
  ▫ Sequences
  ▫ Difference
  ▫ Summation
  ▫ Difference equations
Example of Signal Processing Application

- Compact-Disc (CD) Player
  - Analog sound signals
  - Sampled and stored in digital form
  - Read as digital and converted back to analog
  - High fidelity (Hi-Fi)
Classification of Time-Dependent Signals

- Predictability of their behavior
  - Signals can be random or deterministic
- Variation of their time variable and their amplitude
  - Signals can be either continuous-time or discrete-time
  - Signals can be either analog or discrete amplitude, or digital
- Energy content
  - Signals can be characterized as finite- or infinite-energy signals
- Exhibition of repetitive behavior
  - Signals can be periodic or aperiodic
- Symmetry with respect to the time origin
  - Signals can be even or odd
- Dimension of their support
  - Signals can be of finite or of infinite support. Support
Continuous-Time Signals

- Continuous-amplitude, continuous-time signals are called *analog signals*
- Continuous-amplitude, discrete-time signal is called a *discrete-time signal*
- Discrete-amplitude, discrete-time signal is called a *digital signal*
- If samples of a digital signal are given as binary values, signal is called a *binary signal*
Continuous-Time Signals

- Conversion from continuous to discrete time: *Sampling*
- Conversion from continuous to discrete amplitude: *Quantization* or *Coding*
Continuous-Time Signals

• Example: Speech Signal
Continuous-Time Signals: Examples

- **Example 1:** \( x(t) = \sqrt{2} \cos(\pi t/2 + \pi/4) \) \(-\infty < t < \infty\)
  - Deterministic, analog, periodic, odd, infinite support/energy
- **Example 2:** \( y(t) = (1 + j)e^{j\pi t/2} \) \(0 \leq t \leq 10\)
  - Deterministic, analog, finite support
- **Example 3:** \( p(t) = 1 \) \(0 \leq t \leq 10\)
  - Deterministic, analog, finite support
Basic Signal Operations

- Signal addition
- Constant multiplication
- Time and frequency shifting
  - Shift in time: Delay
  - Shift in frequency: Modulation
- Time scaling
  - Example: $x(-t)$ is a “reflection” of $x(t)$
- Time windowing
  - Multiplication by a window signal $w(t)$
Basic Signal Operations

- **Example:**
  - (a) original signal
  - (b) delayed version
  - (c) advanced version
  - (d) Reflected version

- **Remark:**
  - Whenever we combine the delaying or advancing with reflection, delaying and advancing are swapped
  - Ex 1: $x(-t+1)$ is reflected and delayed
  - Ex 2: $x(-t-1)$ is reflected and advanced
Basic Signal Operations

- Example: Find mathematical expressions for $x(t)$ delayed by 2, advanced by 2, and reflected when:

$$x(t) = \begin{cases} 
1 & 0 \leq t \leq 1 \\
0 & \text{otherwise}
\end{cases}$$

- For delay by 2, replace $t$ by $t-2$

$$x(t - 2) = \begin{cases} 
1 & 0 \leq t - 2 \leq 1 \text{ or } 2 \leq t \leq 3 \\
0 & \text{otherwise}
\end{cases}$$

- For advance by 2

$$x(t + 2) = \begin{cases} 
1 & 0 \leq t + 2 \leq 1 \text{ or } -2 \leq t \leq -1 \\
0 & \text{otherwise}
\end{cases}$$

- For reflection

$$x(-t) = \begin{cases} 
1 & 0 \leq -t \leq 1 \text{ or } -1 \leq t \leq 0 \\
0 & \text{otherwise}
\end{cases}$$
Even and Odd Signals

• Symmetry with respect to the origin

\[
\begin{align*}
x(t) \text{ even} : & \quad x(t) = x(-t) \\
x(t) \text{ odd} : & \quad x(t) = -x(-t)
\end{align*}
\]

• Decomposition of any signal as even/odd parts

\[
\begin{align*}
\gamma(t) &= \gamma_e(t) + \gamma_o(t) \\
\gamma_e(t) &= 0.5 \left[ \gamma(t) + \gamma(-t) \right] \\
\gamma_o(t) &= 0.5 \left[ \gamma(t) - \gamma(-t) \right]
\end{align*}
\]

• Example: \( x(t) = \cos(2\pi t + \theta) \quad -\infty < t < \infty \)
  ▫ Neither even nor odd for \( \theta \neq 0 \) or multiples of \( \pi/2 \)
Periodic and Aperiodic Signals

- Analog signal $x(t)$ is periodic if
  - It is defined for all possible values of $t$, $-\infty < t < \infty$
  - There is a positive real value $T_0$, called the period, such that for some integer $k$, $x(t+kT_0) = x(t)$
- The period is the smallest possible value of $T_0 > 0$ that makes the periodicity possible.
  - Although $NT_0$ for an integer $N > 1$ is also a period of $x(t)$, it should not be considered the period
    - Example: $\cos(2\pi t)$ has a period of 1 not 2 or 3
Periodic and Aperiodic Signals

- Analog sinusoids of frequency $\Omega_0 > 0$ are periodic of period $T_0 = \frac{2\pi}{\Omega_0}$.
  - If $\Omega_0 = 0$, the period is not well defined.
- The sum of two periodic signals $x(t)$ and $y(t)$, of periods $T_1$ and $T_2$, is periodic if the ratio of the periods $T_1/T_2$ is a rational number $N/M$, with $N$ and $M$ being nondivisible.
  - The period of the sum is $MT_1 = NT_2$
- The product of two periodic signals is not necessarily periodic
  - The product of two sinusoids is periodic.
Periodic and Aperiodic Signals

• Example 1
Consider a periodic signal $x(t)$ of period $T_0$. Determine whether the following signals are periodic, and if so, find their corresponding periods:

(a) $y(t) = A + x(t)$.
(b) $z(t) = x(t) + v(t)$ where $v(t)$ is periodic of period $T_1 = NT_0$, where $N$ is a positive integer.
(c) $w(t) = x(t) + u(t)$ where $u(t)$ is periodic of period $T_1$, not necessarily a multiple of $T_0$. Determine under what conditions $w(t)$ could be periodic.

• Example 2
Let $x(t) = e^{j2t}$ and $y(t) = e^{j\pi t}$, and consider their sum $z(t) = x(t) + y(t)$, and their product $w(t) = x(t)y(t)$. Determine if $z(t)$ and $w(t)$ are periodic, and if so, find their periods. Is $p(t) = (1 + x(t))(1 + y(t))$ periodic?
Finite-Energy and Finite-Power Signals

- Concepts of energy and power introduced in circuit theory can be extended to any signal
  - Instantaneous power
  - Energy
  - Power

\[ p(t) = v(t)i(t) = i^2(t) = v^2(t) \]

\[ E_T = \int_{t_0}^{t_1} p(t)dt = \int_{t_0}^{t_1} i^2(t)dt = \int_{t_0}^{t_1} v^2(t)dt \]

\[ P_T = \frac{E_T}{T} = \frac{1}{T} \int_{t_0}^{t_1} i^2(t)dt = \frac{1}{T} \int_{t_0}^{t_1} v^2(t)dt \]
Finite-Energy and Finite Power Signals

- Energy of an analog signal \( x(t) \)
  \[
  E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt
  \]

- Power of an analog signal \( x(t) \)
  \[
  P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt
  \]

- Signal is *finite energy* (or *square integrable*) if \( E_x < \infty \)
- Signal is *finite power* if \( P_x < \infty \)
Finite-Energy and Finite Power Signals: Example

Find the energy and the power of the following:

(a) The periodic signal \( x(t) = \cos(\pi t/2 + \pi/4) \).
(b) The complex signal \( y(t) = (1 + j)e^{j\pi t/2} \), for \( 0 \leq t \leq 10 \) and zero otherwise.
(c) The pulse \( z(t) = 1 \), for \( 0 \leq t \leq 10 \) and zero otherwise.

\[
E_x = \int_{-\infty}^{\infty} \cos^2(\pi t/2 + \pi/4) dt \to \infty
\]

\[
P_x = \frac{1}{8} \int_{0}^{4} \cos(\pi t + \pi/2) dt + \frac{1}{8} \int_{0}^{4} dt = 0 + 0.5 = 0.5
\]

\[
E_y = \int_{0}^{10} |(1 + j)e^{j\pi t/2}|^2 dt = 2 \int_{0}^{10} dt = 20
\]

\[
E_z = \int_{0}^{10} dt = 10
\]

Finite Energy Signals: Zero Power

\[
P_x = \lim_{T \to \infty} \frac{E_x}{2T} = 0
\]
Representation Using Basic Signals

- A fundamental idea in signal processing is to attempt to represent signals in terms of basic signals, which we know how to process
  - Impulse
  - Unit-step
  - Ramp
  - Sinusoids
  - Complex exponentials
Impulse and Unit-Step Signals

- The impulse signal $\delta(t)$ is:
  - Zero everywhere except at the origin where its value is not well defined
  - Its area is equal to unity
- Impulse signal $\delta(t)$ and unit step signal $u(t)$ are related by:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

$$\delta(t) = \frac{du(t)}{dt}$$
The ramp signal is defined as

\[ r(t) = t \ u(t) \]

Its relation to the unit-step and the unit-impulse signals is

\[ \frac{dr(t)}{dt} = u(t) \]

\[ \frac{d^2r(t)}{dt^2} = \delta(t) \]
Sinusoids

Sinusoids are of the general form

\[ A \cos(\Omega_0 t + \theta) = A \sin(\Omega_0 t + \theta + \pi/2) \quad -\infty < t < \infty \]

\[ \Omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} \]
Review of Complex Numbers

• A complex number $z$ represents any point $(x, y)$:
  \[ z = x + jy, \]
  ▫ $x = \text{Re}[z]$ (real part of $z$)
  ▫ $y = \text{Im}[z]$ (imaginary part of $z$)
  ▫ $j = \sqrt{-1}$

• Mathematical representations:
  ▫ Rectangular or polar form: $z = x + jy = |z|e^{j\theta}$
  ▫ Magnitude: $|\bar{z}| = \sqrt{x^2 + y^2} = |z|$ and Phase $\theta = \angle \bar{z} = \angle z$

• Conjugate: $z^* = x - jy = |z|e^{-j\angle z}$
Complex Exponentials

A complex exponential is a signal of the form

\[ x(t) = Ae^{at} \]

\[ = |A|e^{rt} \left[ \cos(\Omega_0 t + \theta) + j \sin(\Omega_0 t + \theta) \right] \quad -\infty < t < \infty \]

where \( A = |A|e^{j\theta} \), and \( a = r + j\Omega_0 \) are complex numbers.

- Depending on the values of \( A \) and \( a \), several signals can be obtained from the complex exponential
Basic Signal Operations—Time Scaling, Frequency Shifting, and Windowing

Given a signal $x(t)$, and real values $\alpha \neq 0$ or $1$, and $\phi > 0$:

- $x(\alpha t)$ is said to be contracted if $|\alpha| > 1$, and if $\alpha < 0$ it is also reflected.
- $x(\alpha t)$ is said to be expanded if $|\alpha| < 1$, and if $\alpha < 0$ it is also reflected.
- $x(t)e^{j\phi t}$ is said to be shifted in frequency by $\phi$ radians.
- For a window signal $w(t)$, $x(t)w(t)$ displays $x(t)$ within the support of $w(t)$. 
Sifting Property

- Property of the impulse function

\[
\int_{-\infty}^{\infty} f(t) \delta(t - \tau) \, dt = \int_{-\infty}^{\infty} f(\tau) \delta(t - \tau) \, dt = f(\tau) \int_{-\infty}^{\infty} \delta(t - \tau) \, dt
\]

\[
= f(\tau) \quad \text{for any } \tau
\]
Problem Assignments

• Problems: 1.4, 1.5, 1.12, 1.13, 1.14, 1.18
• Partial Solutions available from the student section of the textbook web site