Digital Signal Processing – Chapter 10

Fourier Analysis of Discrete-Time Signals and Systems

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Discrete-Time Fourier Transform

- Sampled time domain signal has a Fourier transform that is periodic
- Periodic time domain signal has a Fourier transform that is sampled
- Periodic and sampled time domain signal has a Fourier transform that is both periodic and sampled

Forward DTFT

$$X(e^{j\omega}) = \sum_{n} x[n]e^{-j\omega n} \qquad -\pi \leq \omega < \pi$$

Inverse DTFT

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

Notes on DTFT

- DTFT is just one period of the continuous Fourier transform of the sampled signal
- Z-transform is related to the DTFT as follows:

$$X_s(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$$

  - DTFT is the values on the unit circle in z-Transform
  - Unit circle must be included in the ROC
- Problem: the sampled signal is still infinite in length
  - Impossible to obtain using computers
  - Consider cases when the signal has compact support

Discrete Fourier Transform (DFT)

- Defined as the Fourier transform of a signal that is both discrete and periodic
  - Fourier transform will also be discrete and periodic
  - Can assume periodicity if we have a finite duration of a signal: better than zero assumption since it allows reducing the frequency domain to a sampled function

DFT Formula

- Given a periodic signal $x[n]$ of period N, its DFT is given by,

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \qquad 0 \leq k \leq N-1$$

- Its inverse is given by,

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi nk/N} \qquad 0 \leq n \leq N-1$$

- Both $X[k]$ and $x[n]$ are periodic of the same period N.

Computation of DFT Using FFT

- A very efficient computation of the DFT is done by means of the FFT algorithm, which takes advantage of some special characteristics of the DFT
  - DFT requires O(N2) computations
  - FFT requires $O(N \log_2 N)$ computations
  - Both compute the same thing
- It should be understood that the FFT is not another transformation but an algorithm to efficiently compute DFTs.
- Always remember that the signal and its DFT are both periodic

Improving Resolution in DFT

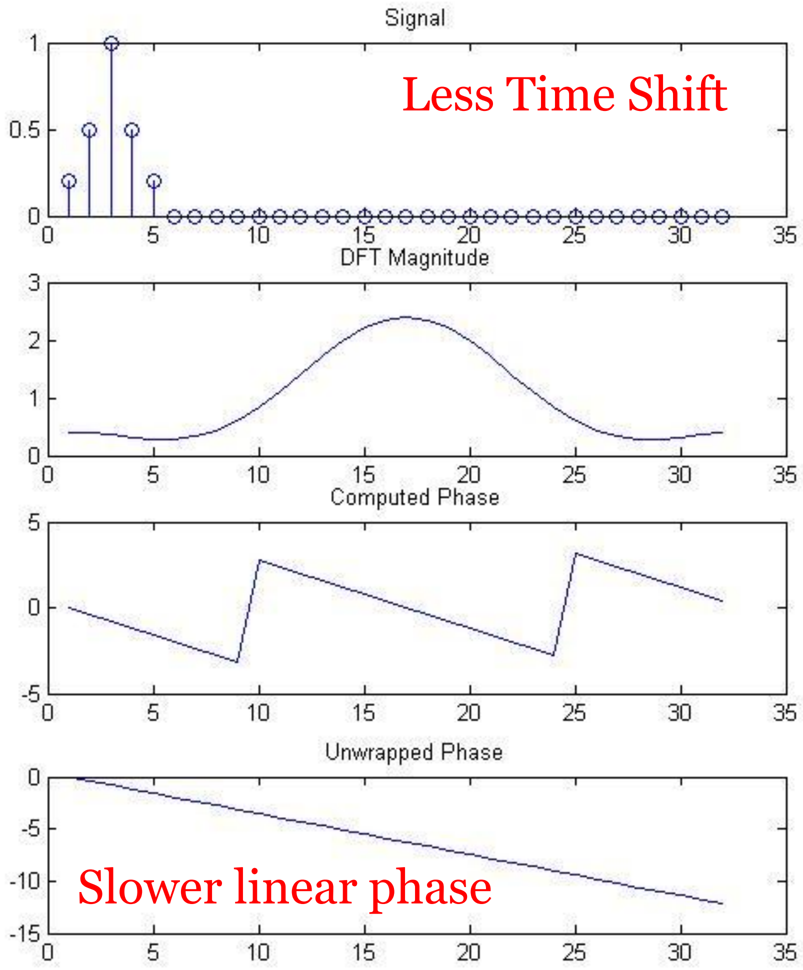
- When the signal $x[n]$ is periodic of period N, the DFT values are normalized Fourier series coefficients of $x[n]$ that only exist for the N harmonic frequencies $\{2\pi k/N\}$, as no frequency components exist for any other frequencies
  - The number of possible frequencies depend on the length L chosen to compute its DFT.
  - Frequencies at which we compute the DFT can be seen as frequencies around the unit circle in the z-plane
- We can improve the frequency resolution of its DFT by increasing the number of samples in the signal without distorting the signal by *padding the signal with zeros*
  - Increase samples of the units circle

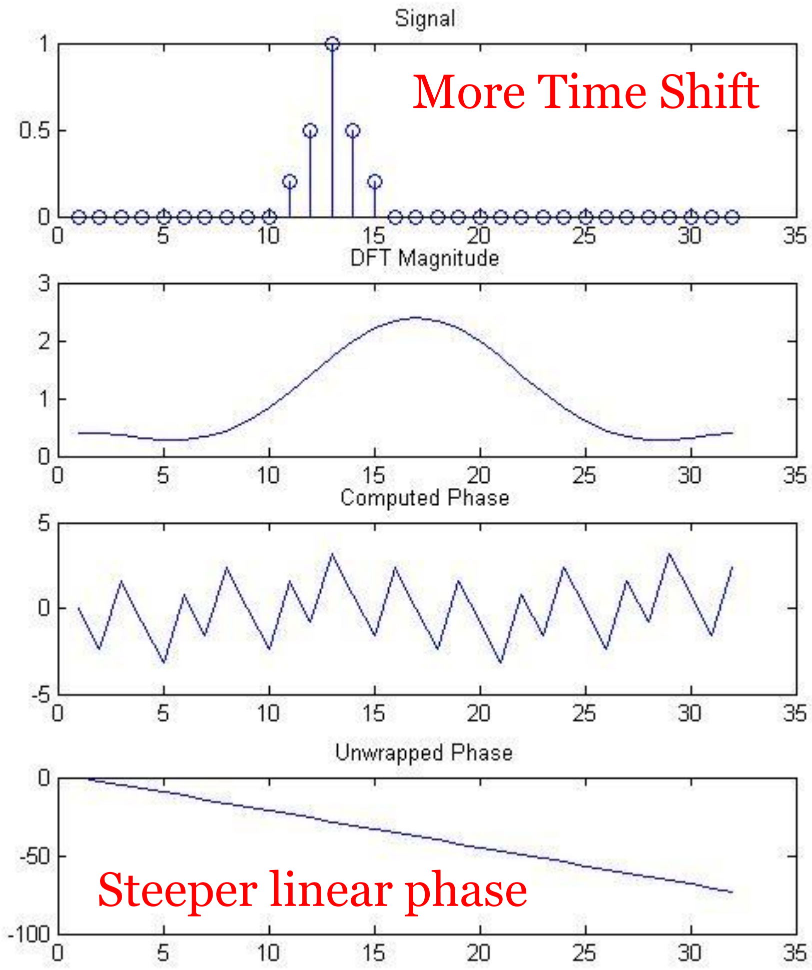
Phase in DFT

- Output of DFT is complex numbers
- Phase is just as important as magnitude in DFT
- Time shift amounts to linear phase
  - Phase wrapping occurs for large shifts
- Linear phase can be unwrapped to its correct form using the “unwrap” function of Matlab

```
N= 32; x= zeros(N,1); x(11:15)= [0.2 0.5 1 0.5 0.2];
X= fftshift(fft(x));
subplot(4,1,1); stem(1:N,x);  title('Signal')
subplot(4,1,2); plot(abs(X));  title('DFT Magnitude')
subplot(4,1,3); plot(angle(X)); title('Computed Phase')
subplot(4,1,4); plot(unwrap(angle(X))); title('Unwrapped Phase')
```

Phase in DFT: Examples

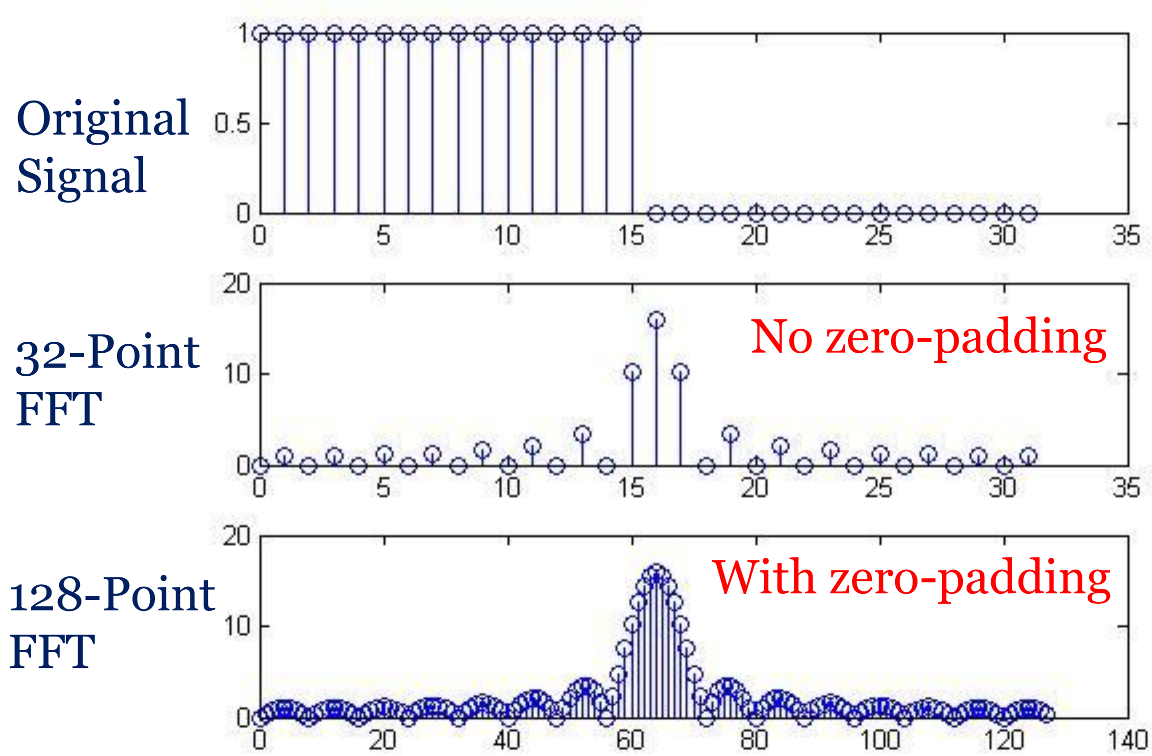




Zero-Padding Example

Matlab Code

```
N= 32;
N2= 128;
n=0:N-1;
n2= 0:N2-1;
x= zeros(N,1);
x(1:16)= 1;
X= fftshift(fft(x));
X2= fftshift(fft(x,N2));
subplot(3,1,1)
stem(n,x)
subplot(3,1,2)
stem(n,abs(X))
subplot(3,1,3)
stem(n2,abs(X2))
```



Linear vs. Circular Convolution

- Convolution in continuous and discrete time aperiodic signals is called *linear convolution*
- Convolution can be calculated as a multiplication in the frequency domain
  - Convolution property of the Fourier transform
- Problem: when using DFT to compute convolution of aperiodic signals, the periodicity assumption results in errors in computing the convolution
  - Signal outside the N points used is not zero: periodic extension of the N points disturb the computation
  - What we compute is called *circular convolution*

Linear vs. Circular Convolution

- Given $x[n]$ and $h[n]$ of lengths M and K, the linear convolution sum $y[n]$ of length $N=(M+K-1)$ can be found by following these three steps:
  - Compute DFTs $X[k]$ and $H[k]$ of length $L \geq N$ for $x[n]$ and $h[n]$ using zero-padding
  - Multiply them to get $Y[k]= X[k]H[k]$.
  - Find the inverse DFT of $Y[k]$ of length L to obtain $y[n]$

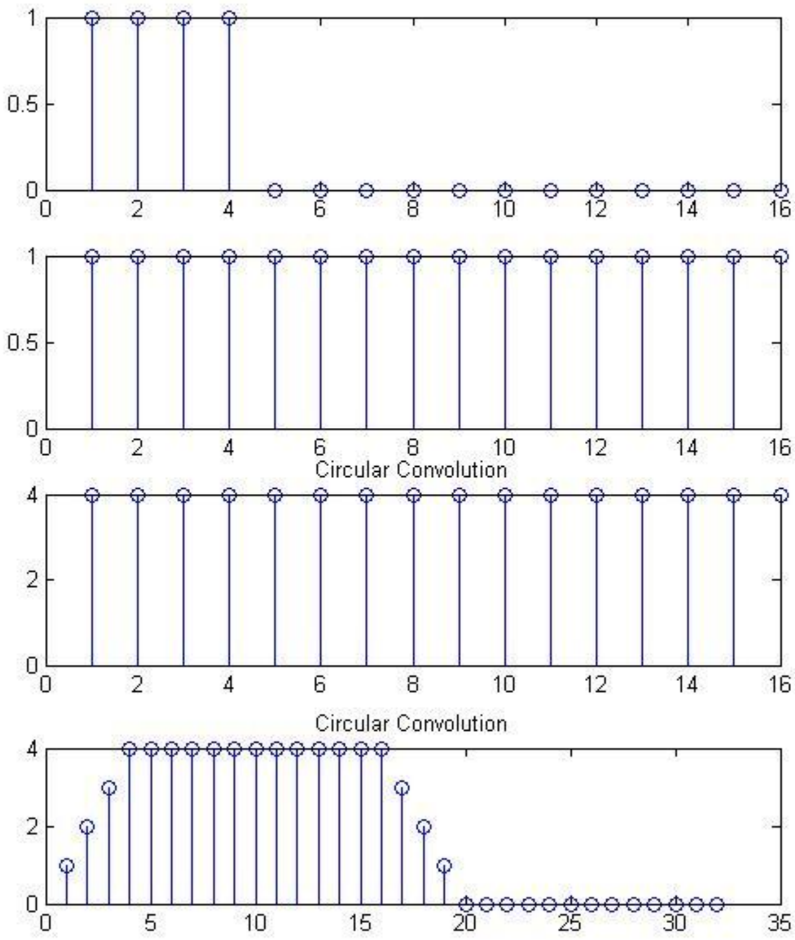
Linear vs. Circular Convolution: Example

```
N1= 16;
N2= 16;
N= 32;   % N>N1+N2-1
x1= zeros(N1,1);
x2= zeros(N2,1);
x1(1:4)= 1;
x2(:)= 1;

 X1= fft(x1,N1);
X2= fft(x2,N1);
X_circ= X1.*X2;
x_circ= ifft(X_circ);

X1= fft(x1,N);
X2= fft(x2,N);
X_lin= X1.*X2;
x_lin= ifft(X_lin);

figure(1); subplot(4,1,1); stem(1:N1,x1)
subplot(4,1,2); stem(1:N2,x2)
subplot(4,1,3); stem(1:N1, abs(x_circ))
title('Circular Convolution');
subplot(4,1,4); stem(1:N,abs(x_lin))
title('Circular Convolution');
```



Fourier Transformations Chart

Signal Classification		Frequency	
		Continuous	Discrete
Time	Continuous	Continuous Fourier Transform (CFT)	Discrete Fourier Series (DFS)
	Discrete	Discrete-Time Fourier Transform (DTFT)	Discrete Fourier Transform (DFT)

Most General Form: others are special cases

Only one that can be done on a computer

Problem Assignments

- Problems: 10.12, 10.26, 10.27, 10.29
- Partial Solutions available from the student section of the textbook web site