Digital Signal Processing – Chapter 11

Introduction to the Design of Discrete Filters

CES Engineering

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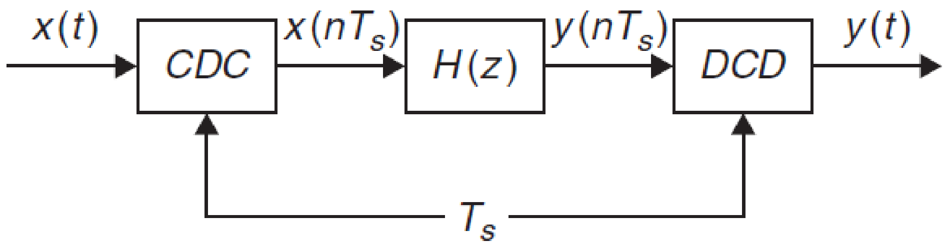
Filters

- Filtering can be seen as a way to change the frequency content of an input signal
- The appropriate filter is specified using the spectral characterization of the input and the desired spectral characteristics of the output of the filter
- Once the specifications of the filter are set, the problem becomes one of approximation to find suitable design and implementation
- Filters can be designed as either Analog or Digital form

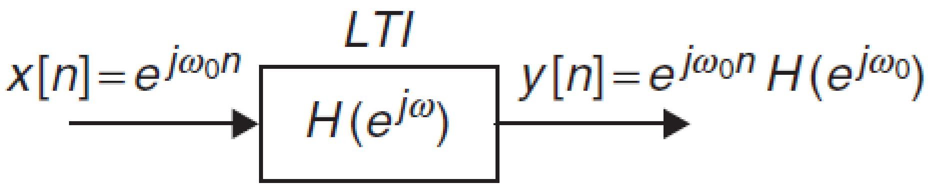


Frequency Selective Discrete Filters

$$Y(z) = H(z)X(z)$$

$$\Downarrow$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$



- By selecting the frequency response $H(e^{j\omega})$ we allow some frequency components of $x[n]$ to appear in the output, and others to be filtered out
- Ideal frequency-selective filters, such as low-pass, high-pass, band-pass, and stopband filters, cannot be realized. They serve as prototypes for the actual filters

Linear Phase

- A filter changes the spectrum of its input in magnitude as well as in phase
  - Distortion in magnitude can be avoided by using an all-pass filter with unit magnitude for all frequencies
  - Phase distortion can be avoided by requiring the phase response of the filter to be linear
- A measure of linearity of the phase is obtained from the **Group Delay** function, which is defined as,

$$\boxed{\tau(\omega) = -\frac{d\theta(\omega)}{d\omega}}$$

  - Group delay is constant when the phase is linear

IIR Discrete Filters

- Infinite-impulse response, recursive or IIR filter has a transfer function in the form:

$$\boxed{H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{m=0}^{M-1} b_m z^{-m}}{1 + \sum_{k=1}^{N-1} a_k z^{-k}} = \sum_{n=0}^{\infty} h[n] z^{-n}}$$

  - Called infinite impulse response since impulse response h[n] typically has infinite length
  - Called recursive because the input–output relationship is given by the difference equation

$$\boxed{y[n] = -\sum_{k=1}^{N-1} a_k y[n-k] + \sum_{m=0}^{M-1} b_m x[n-m]}$$

FIR Discrete Filters

- The transfer function of a finite-impulse response or FIR filter is given as:

$$\boxed{H(z) = B(z) = \sum_{m=0}^{M-1} b_m z^{-m}}$$

  - Its impulse response h[n] has a maximum of M nonzero points, and zero elsewhere, thus of finite length
  - Called Non-recursive because its input-output relationship has the form:

$$\boxed{y[n] = \sum_{m=0}^{M} b_m x[n-m] = (b * x)[n]}$$

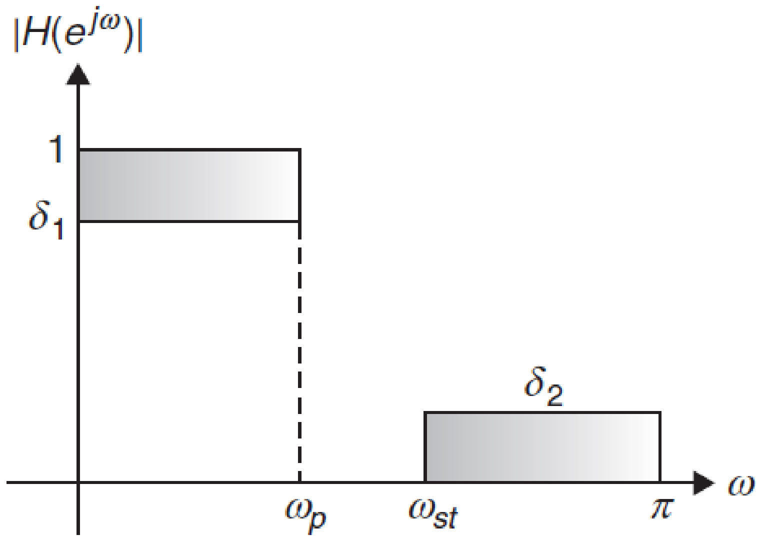
Comparison between IIR and FIR Filters

- *Comparing the IIR and the FIR filters, neither has a definite advantage:*
  - *IIR filters are implemented more efficiently than FIR filters in terms of number of operations and required storage*
  - *Implementation of an IIR filter using the difference equation resulting from its transfer function is simple and computationally efficient, while FIR filters can be implemented using FFT, which is also very efficient*
  - *FIR filters are always BIBO stable, but for an IIR filter we need to check that its poles are inside the unit circle*
  - *FIR filters can be designed to have linear phase, while IIR filters usually have nonlinear phase, but approximately linear phase in passband*

Filter Specifications: Frequency Domain (Linear Scale)

- Magnitude specifications of a discrete low-pass filter in a linear scale are

Passband:	$\delta_1 \leq \lvert H(e^{j\omega})\rvert \leq 1$	$0 \leq \omega \leq \omega_p$
Stopband:	$0 < \lvert H(e^{j\omega})\rvert \leq \delta_2$	$\omega_{st} \leq \omega \leq \pi$

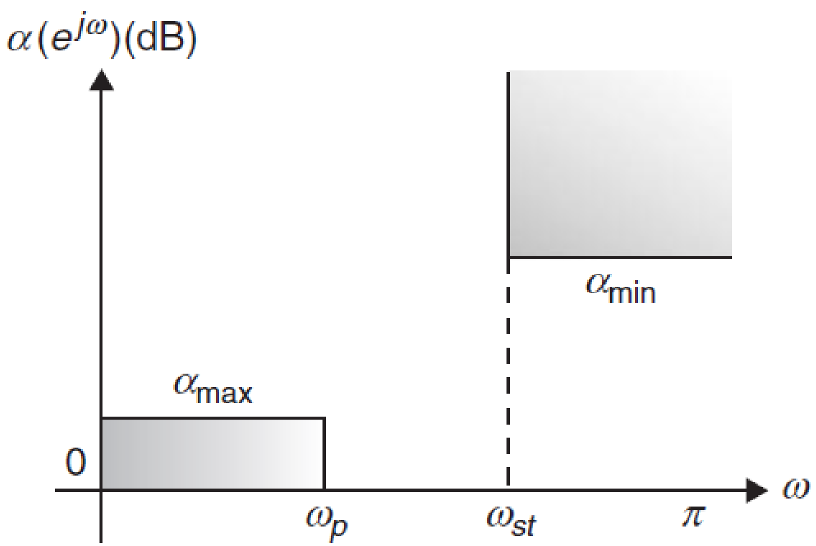


Filter Specifications: Frequency Domain (Log Scale)

- Define the loss function for a discrete filter in log scale as:

$$\boxed{\alpha(e^{j\omega}) = -10\log_{10}|H(e^{j\omega})|^2 = -20\log_{10}|H(e^{j\omega})| \quad \text{dB}}$$

Passband:	$0 \le \alpha(e^{j\omega}) \le \alpha_{\max}$	$0 \le \omega \le \omega_p$
Stopband:	$\alpha_{\min} \le \alpha(e^{j\omega}) < \infty$	$\omega_{st} \le \omega \le \pi$



$$\alpha_{\max} = -20\log_{10}\delta_1$$

$$\alpha_{\min} = -20\log_{10}\delta_2$$

Frequency Scale

- Different equivalent ways in which the frequency of a discrete filter can be expressed:

- The f (Hz) scale from 0 to $f_s/2$, the foldover or Nyquist frequency, that comes from the sampling theory.
- The scale $\Omega = 2\pi f$ (rad/sec) where f is the previous scale, the frequency range is then from 0 to $\Omega_s/2$.
- The discrete frequency scale $\omega = \Omega T_s$ (rad) ranging from 0 to π.
- A normalized discrete-frequency scale ω/π (no units) ranging from 0 to 1.

If the specifications are in the discrete domain, the scale is the ω (rad) or the normalized ω/π.

0	f_p	f_{st}	$0.5\, f_s$	f (Hz)
0	Ω_p	Ω_{st}	$0.5\, \Omega_s$	Ω (rad/sec)
0	ω_p	ω_{st}	π	ω (rad)
0	$\frac{\omega_p}{\pi}$	$\frac{\omega_{st}}{\pi}$	1	$\frac{\omega}{\pi}$

Transformation Design of IIR Discrete Filters

- take advantage of well-understood analog filter design, a common practice is to design discrete filters by means of analog filters and mappings of the *s*-plane into the *z*-plane
- Two mappings used are:
  - The sampling transformation $\boxed{z = e^{sT_s}}$
  - The bilinear transformation,

$$\boxed{s = K\frac{1 - z^{-1}}{1 + z^{-1}}}$$

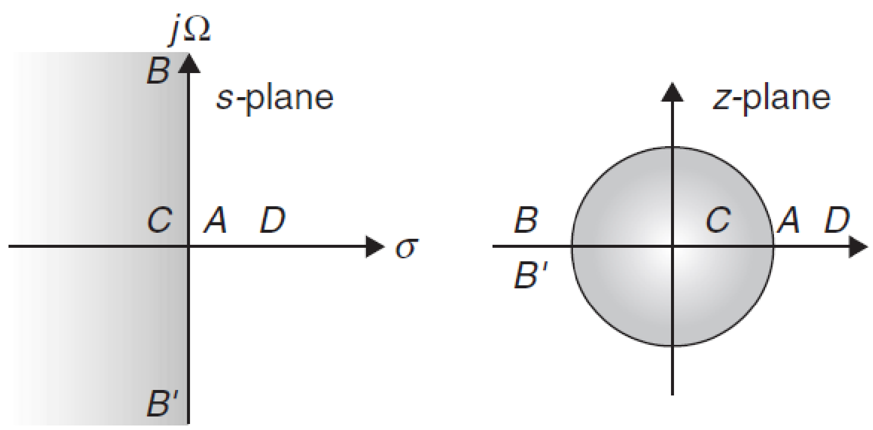
Sampling Transformation Method

- Using this transformation, we convert the analog impulse response $h_a(t)$ of an analog filter into the impulse response $h[n]$ of a discrete filter and obtain the corresponding transfer function
- Advantages:
  - It preserves the stability of the analog filter
  - Given the linear relation between the analog and the discrete frequencies the specifications for the discrete filter can be easily transformed into the specifications for the analog filter
- Resulting design procedure is called: impulse-invariant method

The Bilinear Transformation

$$z\text{- to } s\text{-plane}: \quad s = K\frac{1-z^{-1}}{1+z^{-1}} \qquad K = \frac{2}{T_s}$$

$$s\text{- to } z\text{-plane}: \quad z = \frac{1+s/K}{1-s/K}$$



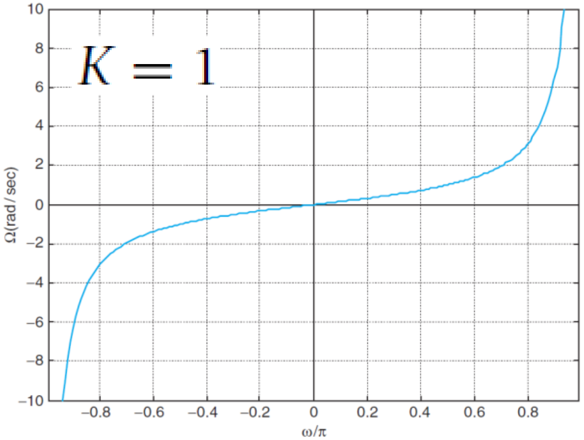
maps

- The $j\Omega$ axis in the s-plane into the unit circle in the z-plane.
- The open left-hand s-plane $\mathcal{R}e[s] < 0$ into the inside of the unit circle in the z-plane, or $|z| < 1$.
- The open right-hand s-plane $\mathcal{R}e[s] > 0$ into the outside of the unit circle in the z-plane, or $|z| > 1$.

Frequency Warping in Bilinear Transformation Method

- A minor drawback of the bilinear transformation is the nonlinear relation between the analog and the
- discrete frequencies
  - Creates a "warping" that needs care when specifying the analog filter using the discrete filter specifications
- The analog frequency Ω and the discrete frequency ω according to the bilinear transformation are related by

$$\boxed{\Omega = K \tan(\omega/2)}$$



FIR Filter Design: Window Design Method

- The usual filter specifications of magnitude and linear phase can be translated into a time-domain specification (i.e., a desired impulse response) by means of the discrete-time Fourier transform then truncated

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\, d\omega \quad \Rightarrow \quad H_d(z) = \sum_{n=-\infty}^{\infty} h_d[n]z^{-n}$$

FIR filter can be obtained as:

$$\boxed{h_w[n] = h_d[n]w[n] = \begin{cases} h_d[n] & -(N-1)/2 \le n \le (N-1)/2 \\ 0 & \text{elsewhere} \end{cases}}$$

where $w[n] = \begin{cases} 1 & -(N-1)/2 \le n \le (N-1)/2 \\ 0 & \text{otherwise} \end{cases}$

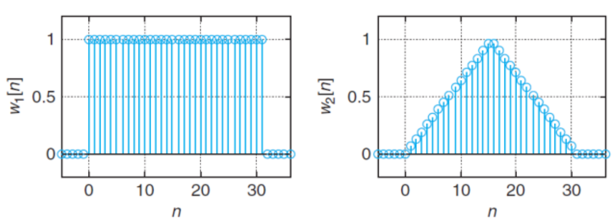
Selection of Windowing Function

Triangular or Barlett window: $\mathrm{w}[n] = \begin{cases} 1 - \frac{2|n|}{N-1} & -(N-1)/2 \le n \le (N-1)/2 \\ 0 & \text{otherwise} \end{cases}$

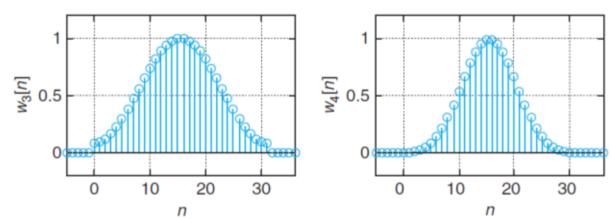
Hamming window: $\mathrm{w}[n] = \begin{cases} 0.54 + 0.46\cos(2\pi n/(N-1)) & -(N-1)/2 \le n \le (N-1)/2 \\ 0 & \text{otherwise} \end{cases}$

Kaiser window: This window has a parameter β that can be adjusted. It is given by

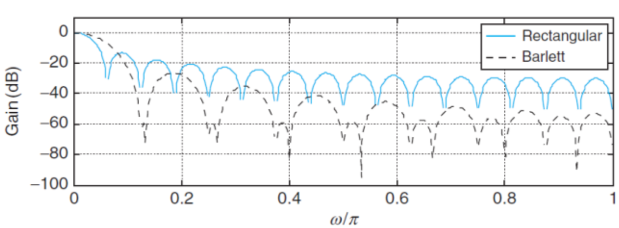
$$\mathrm{w}[n] = \begin{cases} \dfrac{I_0\left(\beta\sqrt{1-(n/(N-1))^2}\right)}{I_0(\beta)} & -(N-1)/2 \le n \le (N-1)/2 \\ 0 & \text{otherwise} \end{cases}$$

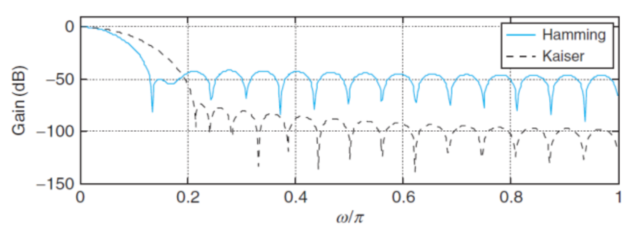


(a) (b)



(a) (b)





FIR Filter Design Example

- Design a low-pass FIR filter with order N=21 to be used in filtering analog signals and that approximates the following ideal frequency response:

$$H_d(e^{j\omega}) = \begin{cases} 1 & 0 \le f \le 125 \text{ Hz} \\ 0 & \text{elsewhere in } 0 \le f \le f_s/2 \end{cases}$$

where $\omega = 2\pi f/f_s$ and $f_s = 1000$ Hz

$$H_d(e^{j\omega}) = \begin{cases} 1 & 0 \le \omega \le \pi/4 \text{ rad} \\ 0 & \text{elsewhere in } 0 \le \omega \le \pi \end{cases}$$

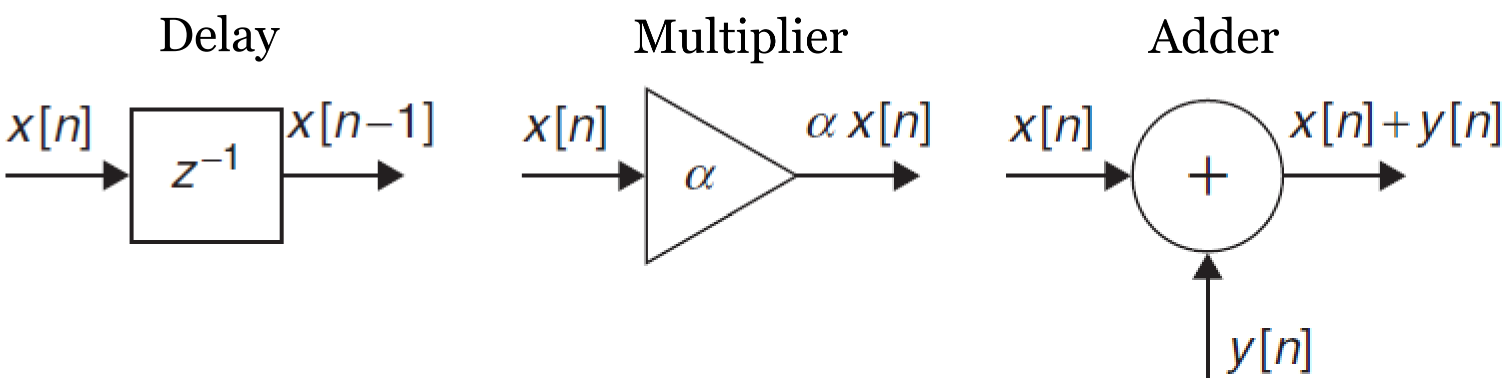
$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega n}d\omega$$

$$= \begin{cases} \sin(\pi n/4)/(\pi n) & n \neq 0 \\ 0.25 & n = 0 \end{cases}$$

$$\boxed{\begin{aligned} \hat{H}(z) &= H_{\mathrm{W}}(z)z^{-10} = \sum_{n=0}^{20} h_d[n-10]z^{-n} \\ &= 0.25z^{-10} + \sum_{n=0, n\neq 10}^{20} \frac{\sin(\pi(n-10)/4)}{\pi(n-10)} z^{-n} \end{aligned}}$$

Realization of Discrete Filters

- Done in hardware or in software
- Requires delays, adders, and constant multipliers

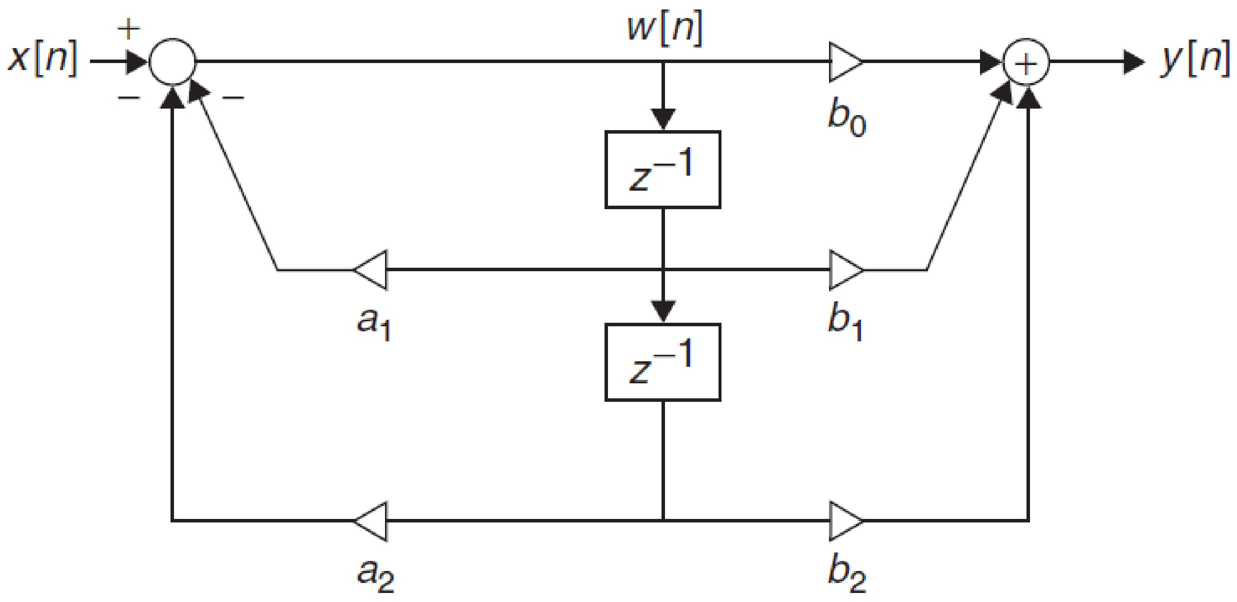


- Two factors for choosing a structure:
  - Computational complexity
  - Quantization effects

Realization of Discrete Filters

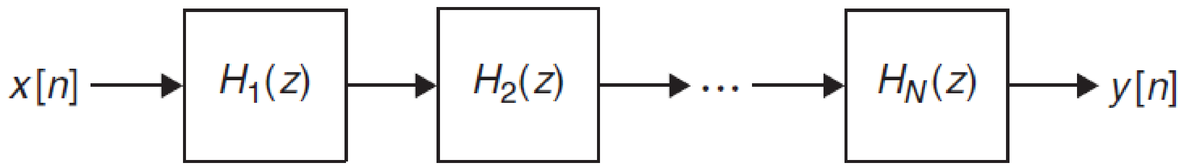
- Direct form:

$$H_2(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$



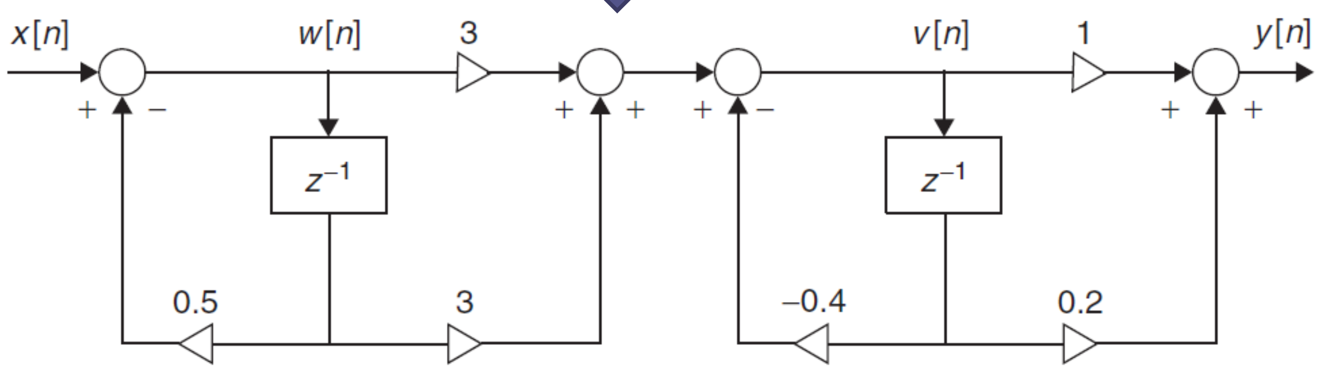
Realization of Discrete Filters

- Cascade Form: $H(z) = \prod H_i(z)$



$H_i(z)$ first- or second-order direct form II realization

$$H(z) = \left[\frac{3(1+z^{-1})}{1+0.5z^{-1}}\right]\left[\frac{1+0.2z^{-1}}{1-0.4z^{-1}}\right]$$

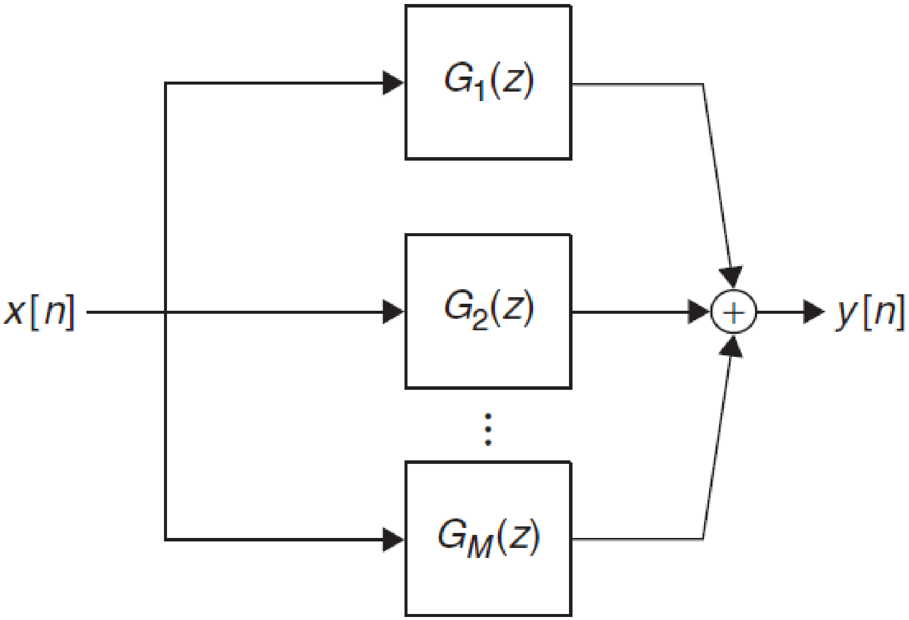


Realization of Discrete Filters

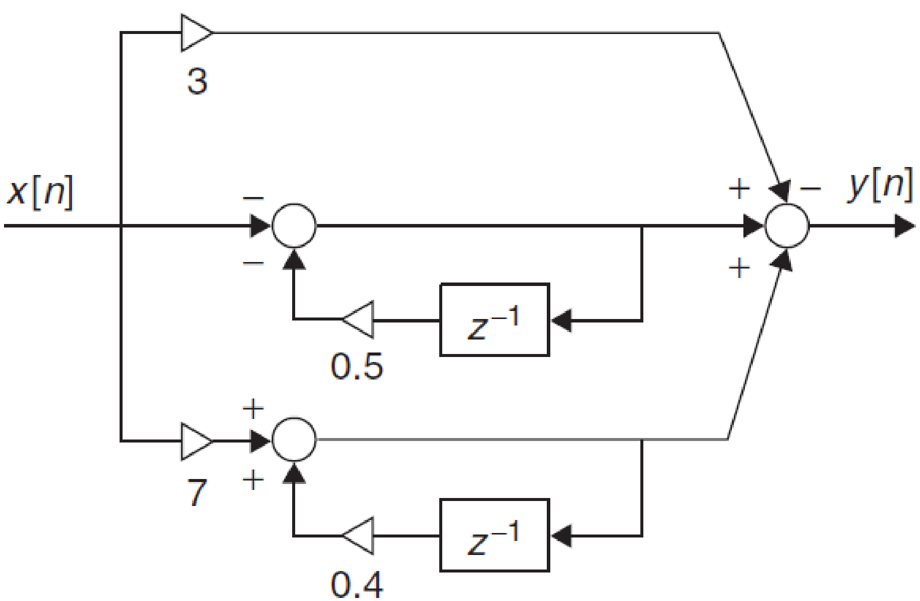
- Parallel Realization: partial fraction expansion

$$H(z) = \frac{B(z)}{A(z)} = C + \sum_{i=1}^{r} H_i(z)$$

$$H(z) = A_1 + \frac{A_2}{1 + 0.5z^{-1}} + \frac{A_3}{1 - 0.4z^{-1}}$$



$G_i(z)$ first- or second-order direct form II realization



Problem Assignments

- Problems: 11.3, 11.4, 11.8, 11.17, 11.18, 11.24, 11.25
- Partial Solutions available from the student section of the textbook web site