Digital Signal Processing – Chapter 7

Sampling Theory

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Sampling Motivation

- Many of the signals found in applications such as communications and control are analog
  - If we wish to process these signals with a computer it is necessary to sample, quantize, and code them to obtain digital signals.
  - Sampling in time + quantization in amplitude
- Sampling issues
  - How to sample in time (sampling period)
  - How to quantize in amplitude (number of bits)

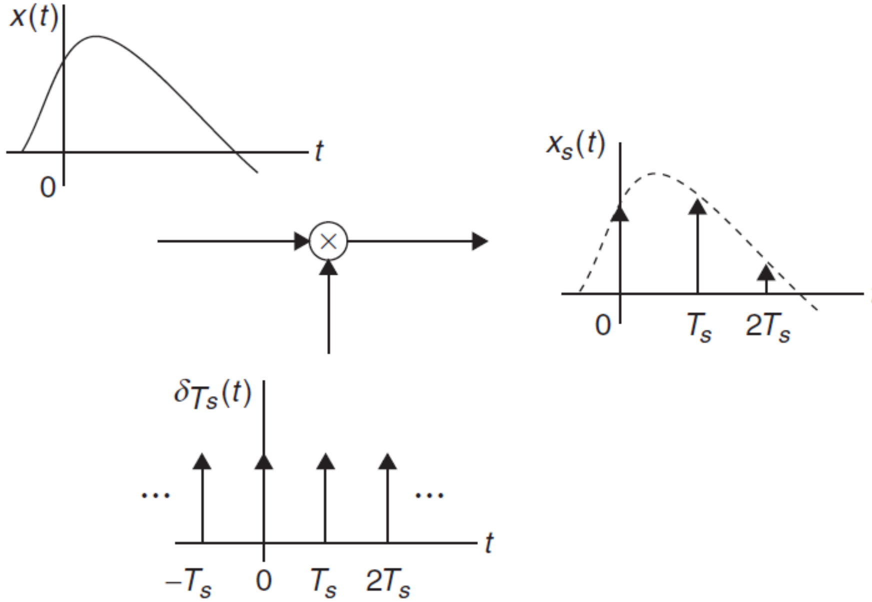
Uniform Sampling

- Sampling function
  - Periodic sequence of impulses of period T_s

$$\delta_{T_s}(t) = \sum_n \delta(t - nT_s)$$

- Sampled signal:

$$x_s(t) = x(t)\delta_{T_s}(t) = \sum_n x(nT_s)\delta(t - nT_s)$$



Uniform Sampling

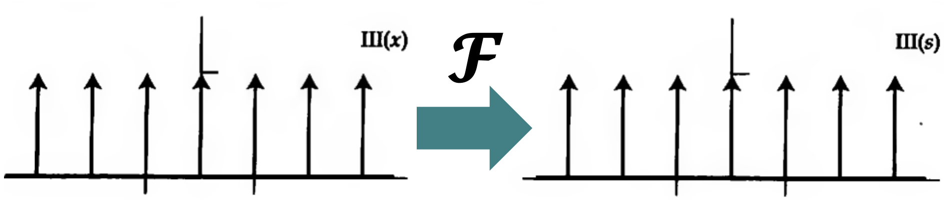
- Fourier Transform of a shifted delta function:

	Function of Time	Function of Ω
1	$\delta(t)$	1
2	$\delta(t-\tau)$	$e^{-j\Omega\tau}$

- Hence,

$$\mathcal{F}\{\sum_n \delta(t-nT_s)\} = \sum_n e^{-j\Omega n T_s} = \sum_m \delta(\Omega - m\frac{2\pi}{T_s})$$

Period = T_s

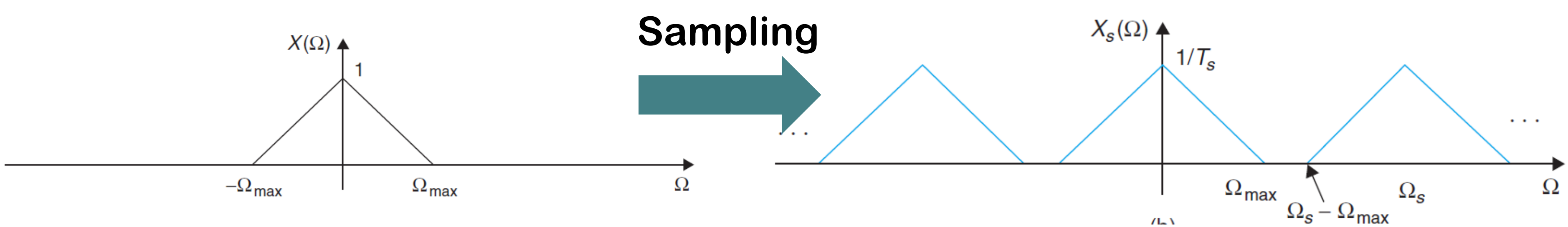


Period $\Omega_s = 2\pi/T_s$

Uniform Sampling

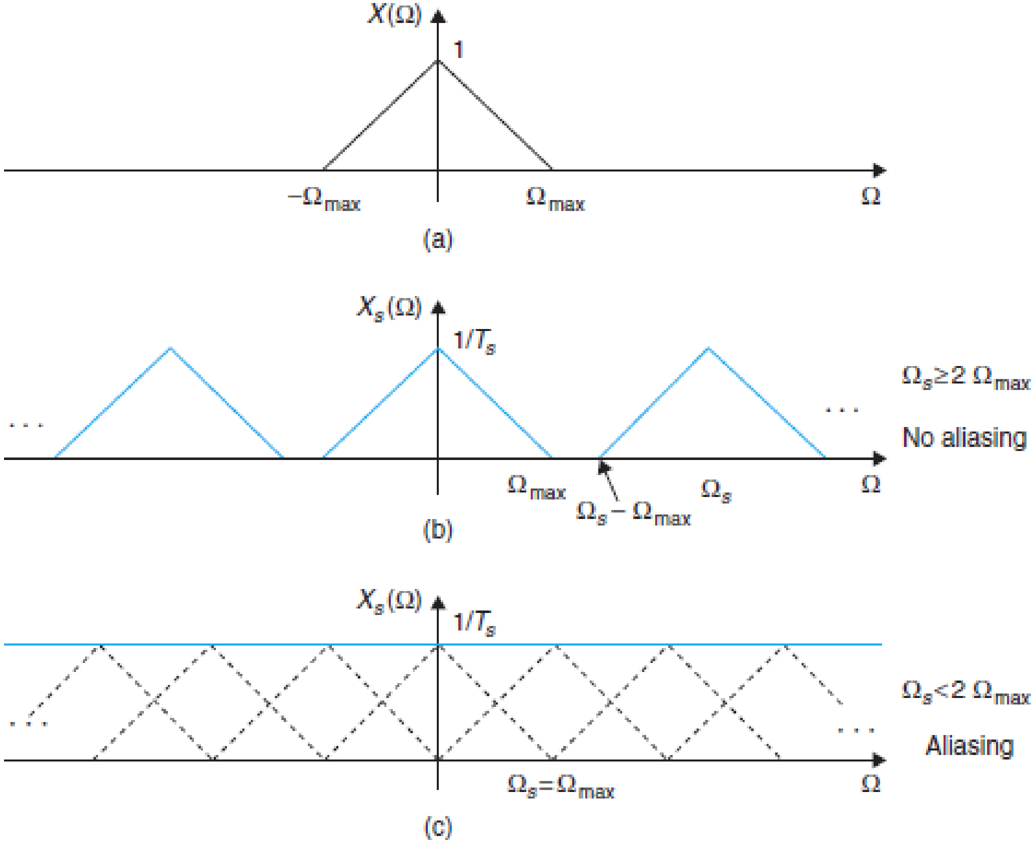
- Sampling in the frequency domain:

$$\mathcal{F}\left\{x(t)\cdot\sum_n \delta(t-nT_s)\right\} = X(\Omega) * \mathcal{F}\left\{\sum_n \delta(t-nT_s)\right\}$$
$$= X(\Omega) * \sum_m \delta(\Omega - m\frac{2\pi}{T_s}) = \sum_m X(\Omega - m\frac{2\pi}{T_s})$$



Sampling Rate Calculation

- Requirement: No Aliasing allowed



Sampling Rate Calculation

- Nyquist criterion

A band-limited signal $x(t)$—that is, its low-pass spectrum $X(\Omega)$ is such that

$$|X(\Omega)| = 0 \text{ for } |\Omega| > \Omega_{\max}$$

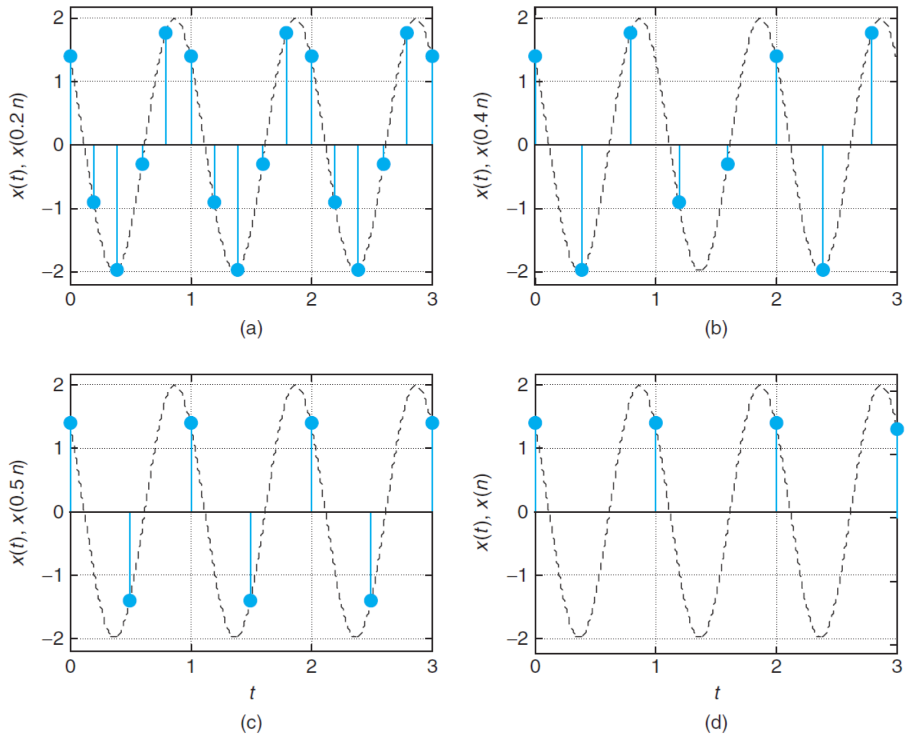
where $\Omega_{\max}$ is the maximum frequency in $x(t)$—can be sampled uniformly and without frequency aliasing using a sampling frequency

$$\Omega_s = \frac{2\pi}{T_s} \geq 2\Omega_{\max}$$

This is called the *Nyquist sampling rate condition*.

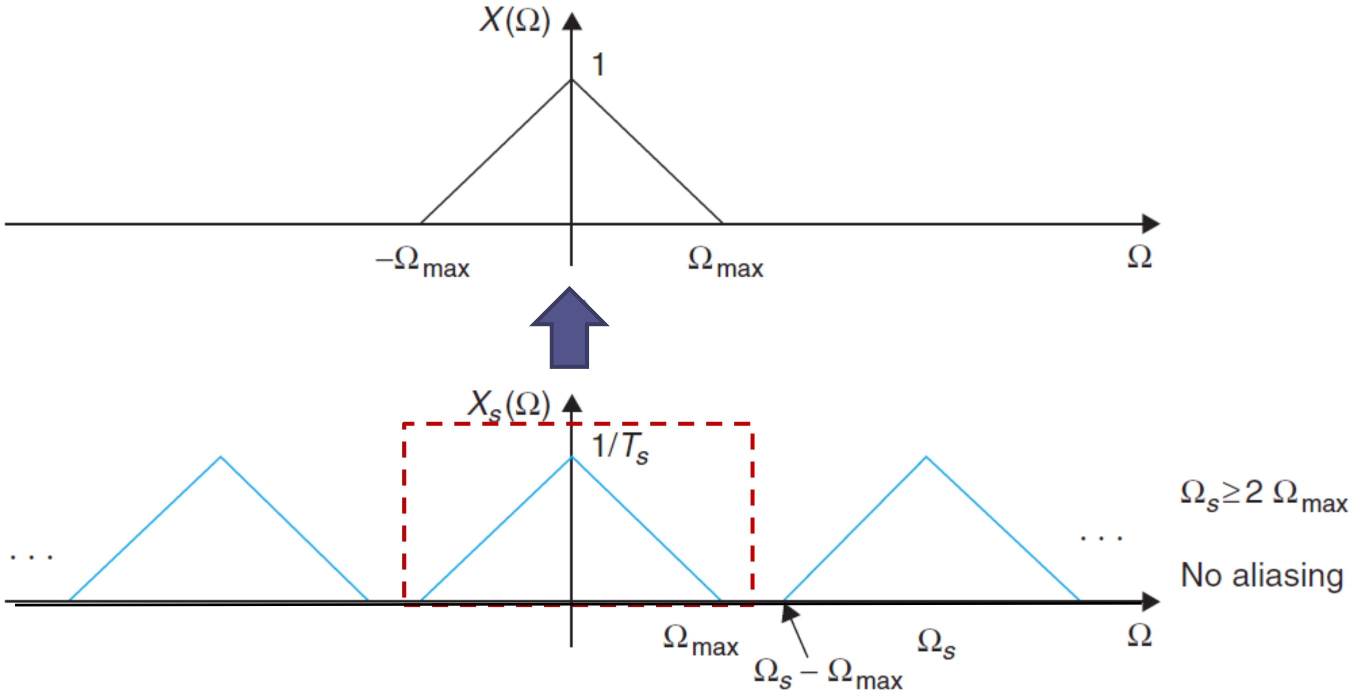
Sampling Rate Calculation: Example

Consider the signal $x(t) = 2\cos(2\pi t + \pi/4)$, $-\infty < t < \infty$. Determine if it is band limited or not. Use $T_s = 0.4$, 0.5, and 1 sec/sample as sampling periods, and for each of these find out whether the Nyquist sampling rate condition is satisfied and if the sampled signal looks like the original signal or not.



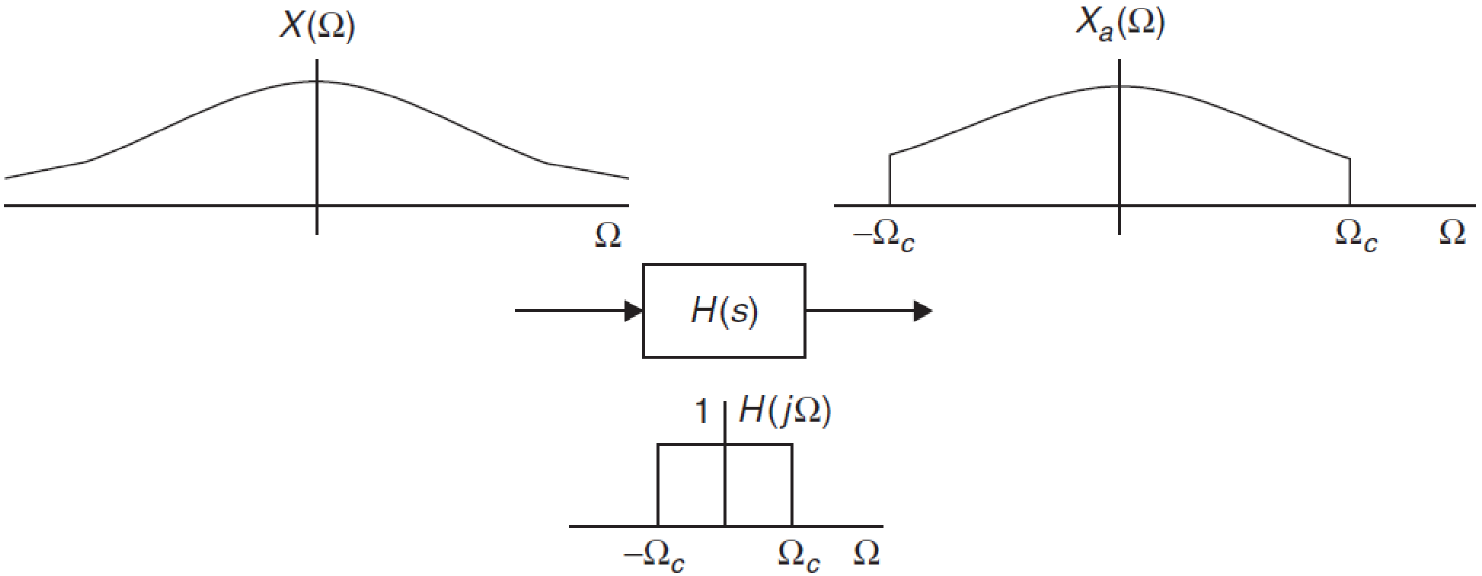
Reconstruction of the Original Continuous-Time Signal from Samples

- Assuming band-limited signal and sufficient sampling, original signal can be recovered by low-pass filtering



Sampling of Infinite Bandwidth Signals: Anti-Aliasing Filter

- Anti-aliasing filter: a low-pass filter applied to the input signal to make sure that the signal to be sampled has a limited bandwidth
  - Applied in all practical analog-to-digital converters



The Nyquist-Shannon Sampling Theorem

If a low-pass continuous-time signal $x(t)$ is band limited (i.e., it has a spectrum $X(\Omega)$ such that $X(\Omega) = 0$ for $|\Omega| > \Omega_{\max}$, where $\Omega_{\max}$ is the maximum frequency in $x(t)$), we then have:

- $x(t)$ is uniquely determined by its samples $x(nT_s) = x(t)|_{t=nT_s}$, $n = 0, \pm 1, \pm 2, \cdots$, provided that the sampling frequency Ω_s (rad/sec) is such that

$\Omega_s \geq 2\Omega_{\max}$ Nyquist sampling rate condition $\Rightarrow$ $f_s = \frac{1}{T_s} \geq \frac{\Omega_{\max}}{\pi}$

- When the Nyquist sampling rate condition is satisfied, the original signal $x(t)$ can be reconstructed by passing the sampled signal $x_s(t)$ through an ideal low-pass filter with the following frequency response:

$$H(\Omega) = \begin{cases} T_s & \frac{-\Omega_s}{2} < \Omega < \frac{\Omega_s}{2} \\ 0 & \text{elsewhere} \end{cases}$$

The reconstructed signal is given by the following sinc interpolation from the samples:

$$x_r(t) = \sum_n x(nT_s) \frac{\sin(\pi(t - nT_s)/T_s)}{\pi(t - nT_s)/T_s}$$

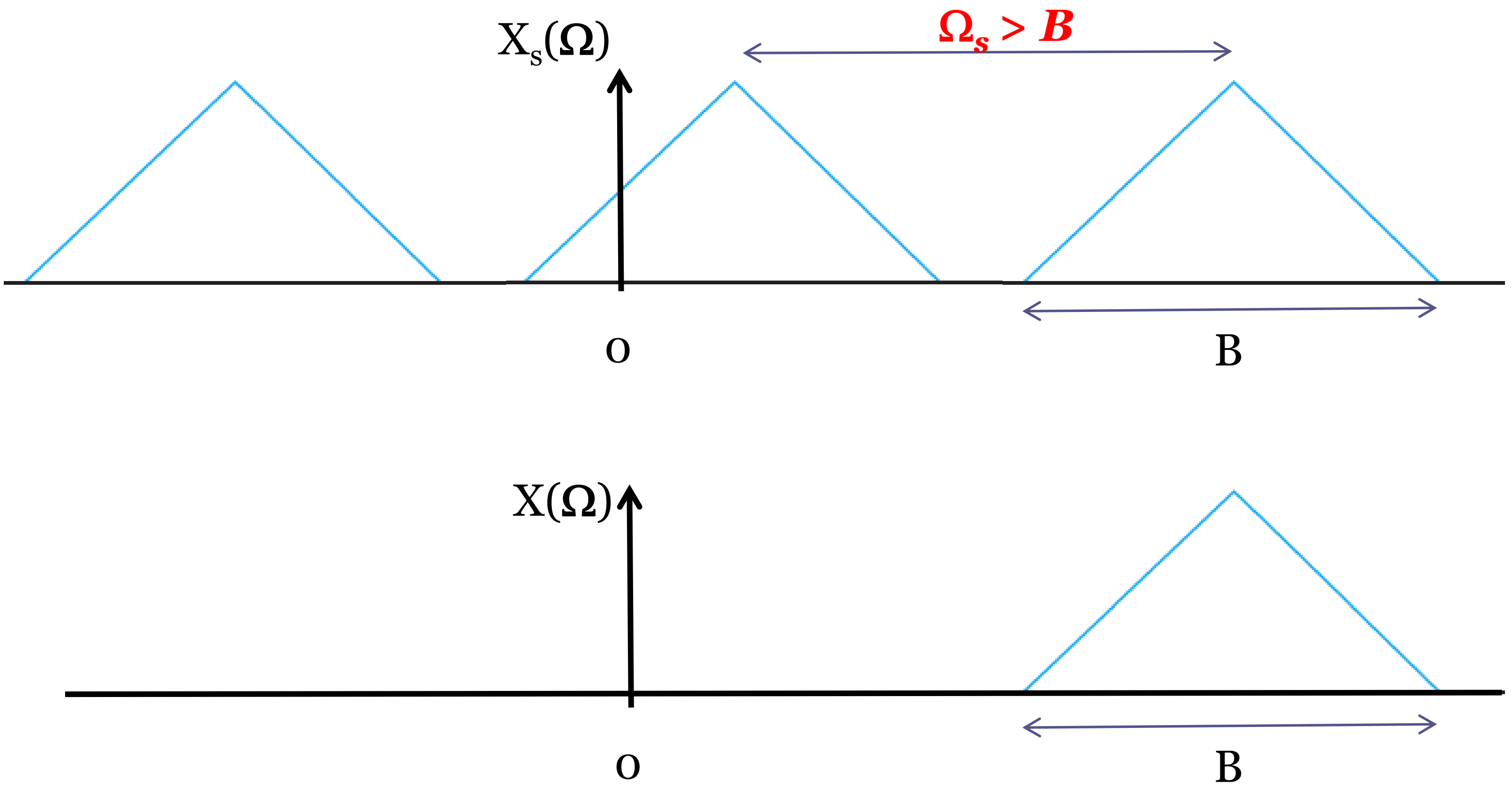
Sampling of Modulated Signals

- Nyquist sampling rate condition applies to low-pass or baseband signals
- Consider a modulated signal $x(t)= m(t)\cos(\Omega_c t)$ where $m(t)$ is the message and $\cos(\Omega_c t)$ is the carrier with carrier frequency $\Omega_c >> \Omega_{max}$
- To avoid aliasing the shifting in frequency should be such that there is no overlapping of the shifted spectra:

$$\Omega_c + \Omega_{\max} - \Omega_s < \Omega_c - \Omega_{\max} \quad \Rightarrow \Omega_s > 2\Omega_{\max}$$

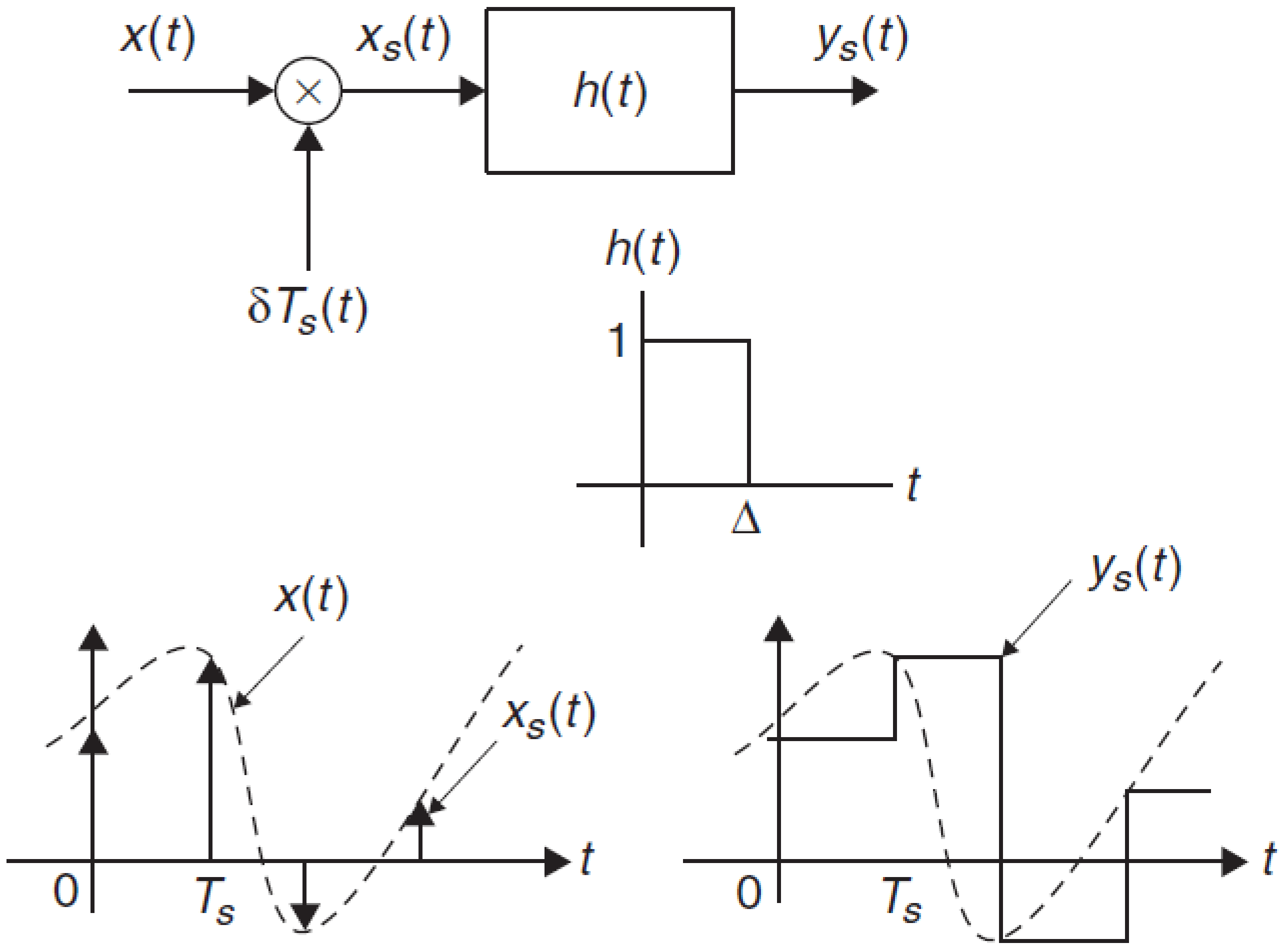
  - Independent of carrier frequency
  - General case: $\boldsymbol{\Omega_s > B}$ *(Bandwidth of signal)*

Sampling of Modulated Signals: Illustration



Practical Aspects: Sample & Hold

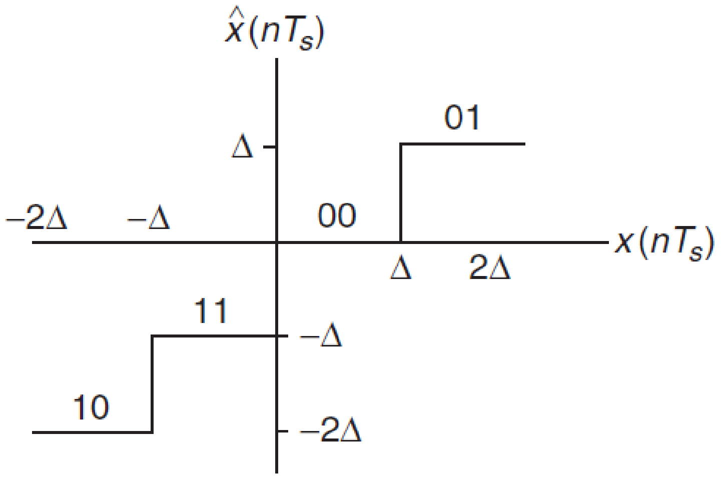
- Samples do not practically have zero duration
  - Time is required for sampling, quantization, and coding



Practical Aspects: Quantization and Coding

- Amplitude discretization of the sampled signal $x_s(t)$ is accomplished by a quantizer consisting of a number of fixed amplitude levels
  - Quantization error

$$\varepsilon(nT_s) = x(nT_s) - \hat{x}(nT_s)$$



Problem Assignments

- Problems: 7.1, 7.2, **7.5**, 7.6, **7.7**, 7.10, 7.11,
- Turn in problems marked in red as homework
- Partial Solutions available from the student section of the textbook web site