Digital Signal Processing – Chapter 8

Discrete-Time Signals and Systems

CES Engineering

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Discrete-Time Signals

- A discrete-time signal x[n] can be thought of as a real- or complex-valued function of the integer sample index n:

$$x[.] : \mathcal{I} \rightarrow \mathcal{R} \ \ (\mathcal{C})$$
$$n \quad x[n]$$

  - For discrete-time signals the independent variable is an integer n, the sample index, and that the value of the signal at n, x[n], is either real or complex
  - Signal is only defined at integer values n—no definition exists for values between the integers
  - Example: sampled signal: $x(nT_s) = x(t)|_{t=nT_s}$

Discrete-Time Signals: Example

- Consider a sinusoidal signal:

$$x(t) = 3\cos(2\pi t + \pi/4) \qquad -\infty < t < \infty$$

Determine an appropriate sampling period T_s and obtain the discrete-time signal $x[n]$ corresponding to the largest allowed sampling period.

- *Solution:*

To sample x(t) so that no information is lost, the Nyquist sampling rate condition indicates that the sampling period shoul

$$T_s \le \frac{\pi}{\Omega_{\max}} = \frac{\pi}{2\pi} = 0.5 \quad \Rightarrow \quad T_s^{\max} = 0.5$$

$$\Rightarrow \quad x[n] = 3\cos(2\pi t + \pi/4)|_{t=0.5n} = 3\cos(\pi n + \pi/4) \qquad -\infty < n < \infty$$

Periodic and Aperiodic Signals

A discrete-time signal $x[n]$ is *periodic* if

- It is defined for all possible values of n, $-\infty < n < \infty$.
- There is a positive integer N, the period of $x[n]$, such that

$$x[n + kN] = x[n]$$

for any integer k.

Periodic discrete-time sinusoids, of period N, are of the form

$$x[n] = A\cos\left(\frac{2\pi m}{N}n + \theta\right) \qquad -\infty < n < \infty$$

where the discrete frequency is $\omega_0 = 2\pi m/N$ rad, for positive integers m and N, which are not divisible by each other, and θ is the phase angle.

Periodic and Aperiodic Signals: Example 1

- Consider the discrete sinusoids:

$$x_1[n] = 2\cos(\pi n - \pi/3)$$

$$x_2[n] = 3\sin(3\pi n + \pi/2) \qquad -\infty < n < \infty$$

⇒ $\omega_1 = \pi = \frac{2\pi}{2}$ ⇒ $m = 1$ and $N = 2$ ⇒ periodic of period $N_1 = 2$

⇒ $\omega_2 = 3\pi = \frac{2\pi}{2}3$ ⇒ $m = 3$ and $N = 2$ ⇒ periodic of period $N_2 = 2$

Periodic and Aperiodic Signals: Example 2

- Continuous-time sinusoids are always periodic but this is not true for discrete-time sinusoids
- Consider: $x[n] = \cos(n + \pi/4)$

The sampled signal $x[n] = x(t)|_{t=nT_s} = \cos(n + \pi/4)$ has a discrete frequency $\omega = 1$ rad that cannot be expressed as $2\pi m/N$ for any integers m and N because π is an irrational number. So $x[n]$ is not periodic.

Since the frequency of the continuous-time signal $x(t)$ is $\Omega = 1$ (rad/sec), then the sampling period, according to the Nyquist sampling rate condition, should be

$$T_s \leq \frac{\pi}{\Omega} = \pi$$

and for the sampled signal $x(t)|_{t=nT_s} = \cos(nT_s + \pi/4)$ to be periodic of period N or

$$\cos((n+N)T_s + \pi/4) = \cos(nT_s + \pi/4) \quad \text{is necessary that} \quad NT_s = 2k\pi$$

$$\Rightarrow \boxed{T_s = 2k\pi/N \leq \pi}$$

Sampling Analog Periodic Signal

When sampling an analog sinusoid

$$x(t) = A\cos(\Omega_0 t + \theta) \qquad -\infty < t < \infty$$

of period $T_0 = 2\pi/\Omega_0$, $\Omega_0 > 0$, we obtain a *periodic discrete sinusoid*,

$$x[n] = A\cos(\Omega_0 T_s n + \theta) = A\cos\left(\frac{2\pi T_s}{T_0}n + \theta\right)$$

provided that

$$T_s \le \frac{\pi}{\Omega_0} = \frac{T_0}{2} \qquad\qquad \frac{T_s}{T_0} = \frac{m}{N}$$

Sum of Discrete-Time Period Signals

- The sum $z[n] = x[n] + y[n]$ of periodic signals $x[n]$ with period N_1, and $y[n]$ with period N_2 is periodic if the ratio of periods of the summands is rational—that is,

$$\frac{N_2}{N_1} = \frac{p}{q}$$

  - Here p and q are integers not divisible by each other
  - If so, the period of $z[n]$ is $qN_2 = pN_1$
- Example: z[n]= sin(πn+2) + cos(2πn/3+1)
  - N1= 2, N2=3 and hence, sum is periodic with period 6
- Example: z[n]= sin(πn+2) + cos(2n/3+1)
  - N1=1, signal 2 is not periodic: sum is not periodic

Finite Energy and Finite Power Discrete-Time Signals

For a discrete-time signal $x[n]$, we have the following definitions:

$$\textit{Energy:} \qquad \varepsilon_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$\textit{Power:} \qquad P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- $x[n]$ is said to have *finite energy* or to be *square summable* if $\varepsilon_x < \infty$.
- $x[n]$ is called *absolutely summable* if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- $x[n]$ is said to have *finite power* if $P_x < \infty$.

Time Shifting, Scaling, and Even/Odd Discrete-Time Signals

A discrete-time signal $x[n]$ is said to be

- *Delayed* by N (an integer) samples if $x[n-N]$ is $x[n]$ shifted to the right N samples.
- *Advanced* by M (an integer) samples if $x[n+M]$ is $x[n]$ shifted to the left M samples.
- *Reflected* if the variable n in $x[n]$ is negated (i.e., $x[-n]$).

Even and odd discrete-time signals are defined as

$$x[n] \text{ is } \textit{even:} \quad \Leftrightarrow \quad x[n] = x[-n]$$

$$x[n] \text{ is } \textit{odd:} \quad \Leftrightarrow \quad x[n] = -x[-n]$$

Any discrete-time signal $x[n]$ can be represented as the sum of an even and an odd component,

$$x[n] = \underbrace{\frac{1}{2}\left(x[n] + x[-n]\right)}_{x_e[n]} + \underbrace{\frac{1}{2}\left(x[n] - x[-n]\right)}_{x_o[n]}$$

$$= x_e[n] + x_o[n]$$

Even/Odd: Example

Find the even and the odd components of the discrete-time signal

$$x[n] = \begin{cases} 4-n & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$x_e[n] = 0.5\big(x[n] + x[-n]\big) \Rightarrow x_e[n] = \begin{cases} 2+0.5n & -4 \le n \le -1 \\ 4 & n = 0 \\ 2-0.5n & 1 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$x_o[n] = 0.5\big(x[n] - x[-n]\big) \Rightarrow x_o[n] = \begin{cases} -2-0.5n & -4 \le n \le -1 \\ 0 & n = 0 \\ 2-0.5n & 1 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Discrete-Time Unit-Step and Unit-Sample Signals

The unit-step $u[n]$ and the unit-sample $\delta[n]$ discrete-time signals are defined as

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

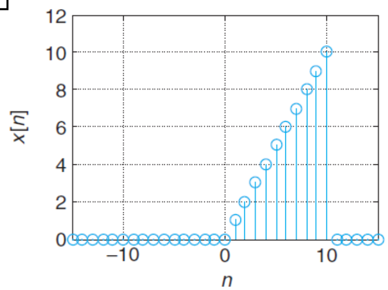
These two signals are related as follows:

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{m=-\infty}^{n} \delta[m]$$

Any discrete-time signal $x[n]$ is represented using unit-sample signals as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



Discrete-Time Systems

- Just as with continuous-time systems, a discrete-time system is a transformation of a discrete-time input signal $x[n]$ into a discrete-time output signal $y[n]$:

$$y[n] = \mathcal{S}\{x[n]\}$$

A discrete-time system $\mathcal{S}$ is said to be

- *Linear:* If for inputs $x[n]$ and $v[n]$ and constants a and b, it satisfies the following
  - *Scaling:* $\mathcal{S}\{ax[n]\} = a\mathcal{S}\{x[n]\}$
  - *Additivity:* $\mathcal{S}\{x[n] + v[n]\} = \mathcal{S}\{x[n]\} + \mathcal{S}\{v[n]\}$

  or equivalently if *superposition* applies—that is,

$$\mathcal{S}\{ax[n] + bv[n]\} = a\mathcal{S}\{x[n]\} + b\mathcal{S}\{v[n]\}$$

- *Time-invariant:* If for an input $x[n]$ with a corresponding output $y[n] = \mathcal{S}\{x[n]\}$, the output corresponding to a delayed or advanced version of $x[n]$, $x[n \pm M]$, is $y[n \pm M] = \mathcal{S}\{x[n \pm M]\}$ for an integer M.

Recursive and Nonrecursive Discrete-Time Systems

- *Recursive system:*

$$y[n] = -\sum_{k=1}^{N-1} a_k y[n-k] + \sum_{m=0}^{M-1} b_m x[n-m] \qquad n \geq 0$$

initial conditions $y[-k],\ k = 1, \ldots, N-1$

This system is also called *infinite-impulse response* (IIR).

- *Nonrecursive system:*

$$y[n] = \sum_{m=0}^{M-1} b_m x[n-m]$$

This system is also called *finite-impulse response* (FIR).

Discrete-Time Systems: Example 1

- **Moving-average discrete filter**: 3rd-order moving-average filter (also called a smoother since it smoothes out the input signal) is an FIR filter for which the input x[n] and the output y[n] are related by:

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

- Linearity: Yes

$$\frac{1}{3}[(ax_1[n] + bx_2[n]) + (ax_1[n-1] + bx_2[n-1]) + (ax_1[n-2] + bx_2[n-2])] = ay_1[n] + by_2[n]$$

- Time Invariance: Yes

$$\frac{1}{3}(x_1[n] + x_1[n-1] + x_1[n-2]) = \frac{1}{3}(x[n-N] + x[n-N-1] + x[n-N-2])$$
$$= y[n-N]$$

Discrete-Time Systems: Example 2

- **Autoregressive discrete filter**: The recursive discrete-time system represented by the first-order difference equation (with initial condition y[-1]):

$$y[n] = ay[n-1] + bx[n] \qquad n \geq 0, \quad y[-1]$$

- **Autoregressive moving average filter**:

$$y[n] = 0.5y[n-1] + x[n] + x[n-1]$$

  - Called the autoregressive moving average given that it is the combination of the two systems

Discrete-Time Systems Represented by Difference Equations

- General form:

$$y[n] = -\sum_{k=1}^{N-1} a_k y[n-k] + \sum_{m=0}^{M-1} b_m x[n-m] \qquad n \geq 0$$

$$\text{initial conditions } \; y[-k], \; k = 1, \ldots, N-1$$

- Just as in the continuous-time case, the system being represented by the difference equation is not LTI unless the initial conditions are zero and the input is causal
- Complete response of a system represented by the difference equation can be shown to be composed of a zero-input and a zero-state responses

$$y[n] = y_{zi}[n] + y_{zs}[n]$$

Discrete Convolution

- For LTI system with impulse response $h[n]$, starting from the generic representation of $x[n]$,

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

We can show that the output can be computed as:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$
$$= \sum_{m=-\infty}^{\infty} x[n-m]h[m]$$

$$[h * x][n] = \sum_{k} x[k]h[n-k] = \sum_{k} x[n-k]h[k]$$
$$= [x * h][n]$$

Note: Convolution is a *linear operator*

Discrete Convolution: Example

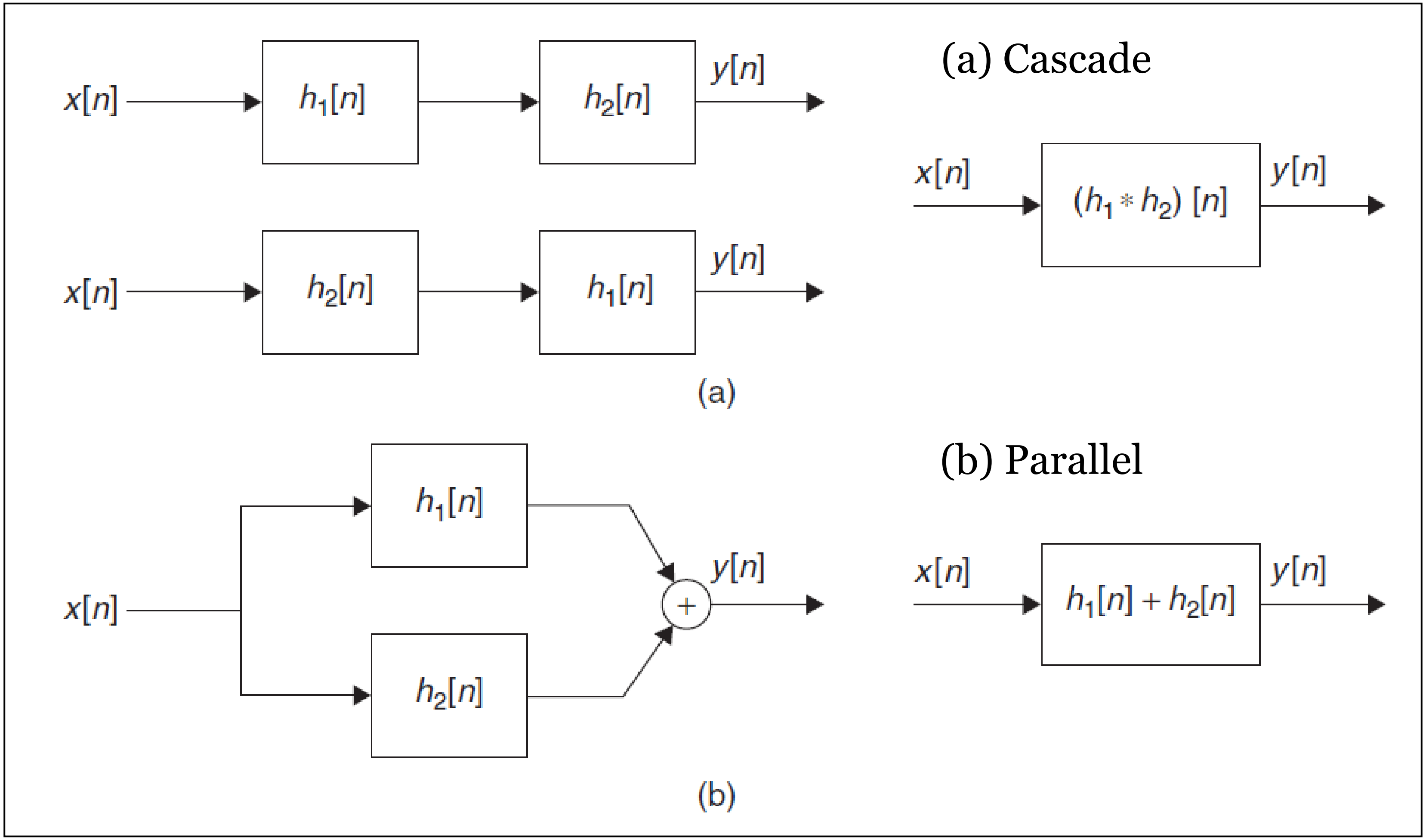
- The output of nonrecursive or FIR systems is the convolution sum of the input and the impulse response of the system:

$$y[n] = \sum_{k=0}^{N-1} b_k x[n-k]$$

- Impulse response is found when $x[n] = \delta[n]$.

$$h[n] = \sum_{k=0}^{N-1} b_k \delta[n-k] = b_0\delta[n] + b_1\delta[n-1] + \cdots + b_{N-1}\delta[n-(N-1)]$$

Cascade and Parallel Connections



Discrete-Time Systems: Example

- Find the impulse response and output for x[n]=u[n] of a moving-averaging filter where the input is x[n] and the output is y[n]:

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

$$\Rightarrow \quad h[n] = \frac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2])$$

$$y[0] = \frac{1}{3}\left(x[0] + x[-1] + x[-2]\right) = \frac{1}{3}x[0]$$

$$y[1] = \frac{1}{3}\left(x[1] + x[0] + x[-1]\right) = \frac{1}{3}(x[0] + x[1])$$

$$y[2] = \frac{1}{3}\left(x[2] + x[1] + x[0]\right) = \frac{1}{3}(x[0] + x[1] + x[2])$$

$$y[3] = \frac{1}{3}\left(x[3] + x[2] + x[1]\right) = \frac{1}{3}(x[1] + x[2] + x[3])$$

$$\cdots$$

Thus, if $x[n]=u[n]$, then:
$y[0]= 1/3$
$y[1] =2/3$
$y[n] =1$ for $n \geq 2$

Causality of Discrete-Time Systems

- A discrete-time system S is causal if:
  - Whenever the input x[n]=0, and there are no initial conditions, the output is y[n]=0.
  - The output y[n] does not depend on future inputs.

- An LTI discrete-time system is *causal* if the impulse response of the system is such that

$$h[n] = 0 \qquad n < 0$$

- A signal $x[n]$ is said to be *causal* if

$$x[n] = 0 \qquad n < 0$$

- For a causal LTI discrete-time system with a causal input $x[n]$ its output $y[n]$ is given by

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k] \qquad n \geq 0$$

Causality: Examples

- Consider the system defined by,

$$y[n] = x^2[n]$$

  - Nonlinear, time invariant and Causal

- Consider the moving average system defined by,

$$y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1]).$$

  - LTI and Non-Causal

Stability of Discrete-Time Systems

- Bounded-Input Bounded-Output (BIBO) Stability
- An LTI discrete-time system is said to be BIBO stable if its impulse response h[n] is absolutely summable:

$$\sum_{k} |h[k]| < \infty$$

- Notes:
  - Nonrecursive or FIR systems are BIBO stable. Indeed, the impulse response of such a system is of finite length and thus absolutely summable.
  - For a recursive or IIR system represented by a difference equation, to establish stability we need to find the system impulse response h[n] and determine whether it is absolutely summable or not.

Stability: Example

- Consider an autoregressive system

$$y[n] = 0.5y[n-1] + x[n]$$

Determine if the system is BIBO stable.

$$h[n] = 0.5^n u[n] \Rightarrow \sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} 0.5^n = \frac{1}{1-0.5} = 2$$

$\Rightarrow$ System is BIBO stable

Problem Assignments

- Problems: 8.1, 8.3, 8.9, 8.10, 8.11, 8.12, 8.17, 8.18
- Partial Solutions available from the student section of the textbook web site