Digital Signal Processing – Chapter 9

The Z-Transform

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Z-Transform

- Just as with the Laplace transform for continuous-time signals and systems, the Z-transform provides a way to represent discrete-time signals and systems, and to process discrete-time signals
  - Although the Z-transform can be related to the Laplace transform, the relation is operationally not very useful
- Representation of discrete-time signals by Z-transform is very intuitive—it converts a sequence of samples into a polynomial
  - As with Laplace transform and convolution integral, the most important property of the Z-transform is the implementation of the convolution sum as a multiplication of polynomials

Laplace Transform of Sampled Signals

- Consider a sampled signal:

$$x(t) = \sum_n x(nT_s)\delta(t - nT_s)$$

- Then,

$$X(s) = \sum_n x(nT_s)\mathcal{L}[\delta(t - nT_s)]$$

$$= \sum_n x(nT_s)e^{-nsT_s}$$

- Let $z = e^{sT_s}$, then this is called the Z-transform of *x(n)* :

$$\mathcal{Z}[x(nT_s)] = \mathcal{L}[x_s(t)]\Big|_{z=e^{sT_s}}$$

$$= \sum_n x(nT_s)z^{-n}$$

Comments About Z-Transform

- Letting s=jΩ, we find that the Fourier transform is a special case when $z=e^{j\Omega}$
  - Periodic Fourier transform since x(t) is sampled
- While Laplace transform may have an infinite number of poles or zeros—complicating the partial fraction expansion when finding its inverse, the inverse Z-transform can be readily obtained using the time-shift property from the *z* polynomial:

$$\mathcal{Z}[x(nT_s)] = \mathcal{L}[x_s(t)]\big|_{z=e^{sT_s}} = \sum_n x(nT_s)z^{-n}$$

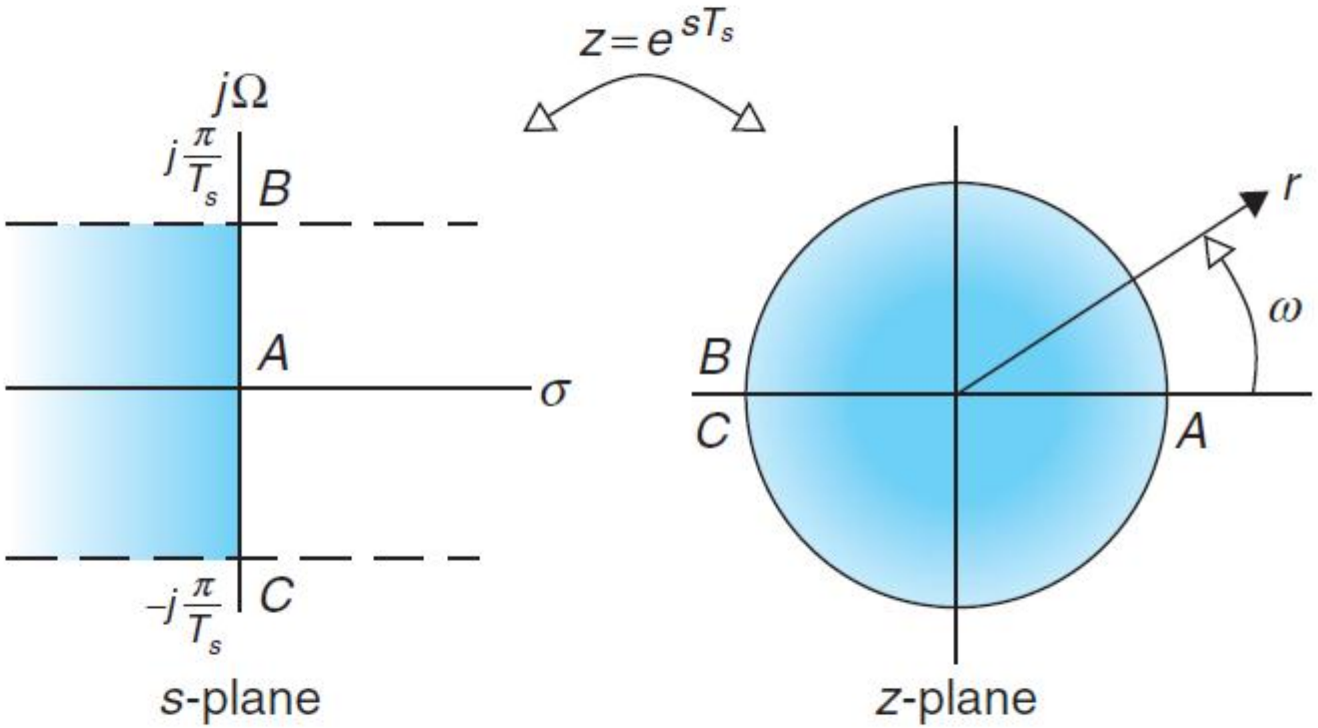
$\Rightarrow$

$$x(t) = \sum_n x(nT_s)\delta(t - nT_s)$$

z-Plane vs. s-Plane

- Connection between the s-plane and the z-plane

$$z = e^{sT_s} = e^{(\sigma + j\Omega)T_s} = e^{\sigma T_s} e^{j\Omega T_s}$$



Forward Z-Transform Definitions

- Two-sided

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

defined in a region of convergence (ROC) in the z-plane.

- One-Sided

$$X_1(z) = \mathcal{Z}(x[n]u[n]) = \sum_{n=0}^{\infty} x[n]u[n]z^{-n}$$

defined in a region of convergence (ROC) in the z-plane.

Region of Convergence

- The infinite summation of the two-sided Z-transform must converge for some values of z
  - For X(z) to converge it is necessary that:

$$|X(z)| = \left|\sum_n x[n] z^{-n}\right| \le \sum_n |x[n]||r^{-n} e^{j\omega n}| = \sum_n |x[n]||r^{-n}| < \infty$$

- Poles and zeros

The poles of a Z-transform $X(z)$ are complex values $\{p_k\}$ such that

$$X(p_k) \to \infty$$

while the zeros of $X(z)$ are the complex values $\{z_k\}$ that make

$$X(z_k) = 0$$

Poles and Zeros: Example

- Find the poles and zeros of the following Z-transforms:

$$\text{(a) } X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

$$\text{(b) } X_2(z) = \frac{(z^{-1} - 1)(z^{-1} + 2)^2}{z^{-1}(z^{-2} + \sqrt{2}z^{-1} + 1)}$$

$$X_1(z) = \frac{z^3(1 + 2z^{-1} + 3z^{-2} + 4z^{-3})}{z^3} = \frac{z^3 + 2z^2 + 3z + 4}{z^3} = \frac{N_1(z)}{D_1(z)}$$

three poles at $z = 0$

zeros are the roots of $N_1(z)$

$$X_2(z) = \frac{z^3(z^{-1} - 1)(z^{-1} + 2)^2}{z^3(z^{-1}(z^{-2} + \sqrt{2}z^{-1} + 1))} = \frac{(1 - z)(1 + 2z)^2}{1 + \sqrt{2}z + z^2} = \frac{N_2(z)}{D_2(z)}$$

poles of $X_2(z)$ are the roots of $D_2(z) = 1 + \sqrt{2}z + z^2 = 0$

zeros of $X_2(z)$ are the roots of $N_2(z) = (1 - z)(1 + 2z)^2 = 0$

ROC of Finite-Support Signals

- The ROC of the Z-transform of a signal x[n] of finite support [N0,N1] where $-\infty < N_0 < n < N_1 < \infty$,

$$X(z) = \sum_{n=N_0}^{N_1} x[n] z^{-n}$$

is the whole z-plane, excluding the origin z= 0 and/or z= ±∞ depending on N_0 and N_1

- Example:

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 9 \\ 0 & \text{otherwise} \end{cases} \Rightarrow X(z) = \sum_{n=0}^{9} 1 \, z^{-n}$$

ROC: Whole z plane except origin

ROC of Infinite-Support Signals

- Signals of infinite support are either causal, anti-causal, or a combination of these or non-causal
- Z-transform of a causal signal $x_c[n]$:

$$X_c(z) = \sum_{n=0}^{\infty} x_c[n] z^{-n} = \sum_{n=0}^{\infty} x_c[n] r^{-n} e^{-jn\omega}$$

- Let R1 be the radius of the farthest-out pole of $X_c(z)$,

$$|X_c(z)| \leq \sum_{n=0}^{\infty} |x_c[n]||r^{-n}| < M \sum_{n=0}^{\infty} \left| \frac{R_1}{r} \right|^n < \infty$$

$$R_1/r < 1 \Rightarrow \boxed{|z| = r > R_1}$$

- Anti-causal $x_a[n]$: ROC is the opposite: $\boxed{|z| = r < R_2}$

ROC of Infinite-Support Signals

- If the signal $x[n]$ is non-causal, it can be expressed as,

$$x[n] = x_c[n] + x_a[n]$$

- ROC: combination of causal and anti-causal ROCs,

$$0 \leq R_1 < |z| < R_2 < \infty$$

For the Z-transform $X(z)$ of an infinite-support signal:

- A causal signal $x[n]$ has a region of convergence $|z| > R_1$ where R_1 is the largest radius of the poles of $X(z)$—that is, the region of convergence is the outside of a circle of radius R_1.
- An anti-causal signal $x[n]$ has as region of convergence the inside of the circle defined by the smallest radius R_2 of the poles of $X(z)$, or $|z| < R_2$.
- A noncausal signal $x[n]$ has as region of convergence $R_1 < |z| < R_2$, or the inside of a torus of inside radius R_1 and outside radius R_2 corresponding to the maximum and minimum radii of the poles of $X_c(z)$ and $X_a(z)$, which are the Z-transforms of the causal and anti-causal components of $x[n]$.

ROC: Example

- Find ROC of the Z-transforms of the following signals:

(a) $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$

$x_1[n]$ is causal

$$X_1(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1 - 0.5z^{-1}} = \frac{z}{z - 0.5}$$

$\Rightarrow$ $\mathcal{R}_1 : |z| > 0.5$

(b) $x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$

$x_2[n]$ is anti-causal.

$$X_2(z) = -\sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n} = -\sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{-m} z^m + 1$$

$$= -\sum_{m=0}^{\infty} 2^m z^m + 1 = \frac{-1}{1 - 2z} + 1 = \frac{z}{z - 0.5}$$

$\Rightarrow$ $\mathcal{R}_2 : |z| < 0.5$

Note: ROC for $x_1[n]+x_2[n]$ is empty: Z-transform does not exist for this sum !!

Linearity and Convolution

The Z-transform is a linear transformation, meaning that

$$\mathcal{Z}(ax[n] + by[n]) = a\mathcal{Z}(x[n]) + b\mathcal{Z}(y[n])$$

for signals $x[n]$ and $y[n]$ and constants a and b.

- Convolution: similar to Laplace and Fourier transforms

$$y[n] = [x * h][n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} h[k]x[n-k]$$

$$Y(z) = \mathcal{Z}\{[x * h][n]\} = \mathcal{Z}\{x[n]\}\mathcal{Z}\{h[n]\} = X(z)H(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\mathcal{Z}[output\ y[n]]}{\mathcal{Z}[input\ x[n]]}$$

Convolution Sum As a Polynomial Multiplication

- Consider $X_1(z) = 1 + a_1 z^{-1} + a_2 z^{-2}$ and $X_2(z) = 1 + b_1 z^{-1}$

$$X_1(z)X_2(z) = 1 + b_1 z^{-1} + a_1 z^{-1} + a_1 b_1 z^{-2} + a_2 z^{-2} + a_2 b_1 z^{-3}$$
$$= 1 + (b_1 + a_1)z^{-1} + (a_1 b_1 + a_2)z^{-2} + a_2 b_1 z^{-3}$$

- The convolution sum of the two sequences [1 a1 a2] and [1 b1], formed by the coefficients of $X_1(z)$ and $X_2(z)$, is given as [1 (a1+b1) (a2+b1 a1) a2], which corresponds to the coefficients of the product of the polynomials $X_1(z)X_2(z)$
- Notice that the sequence of length 3 and the sequence of length 2 when convolved give a sequence of length 3+2-1=4

Finite Impulse Response (FIR) Filter

- A finite-impulse response or FIR filter is implemented by means of the convolution sum
- Consider an FIR with an input–output equation:

$$y[n] = \sum_{k=0}^{N-1} b_k x[n-k]$$

- Impulse response: let $x[n] = \delta[n]$

$$h[n] = \sum_{k=0}^{N-1} b_k \delta[n-k]$$

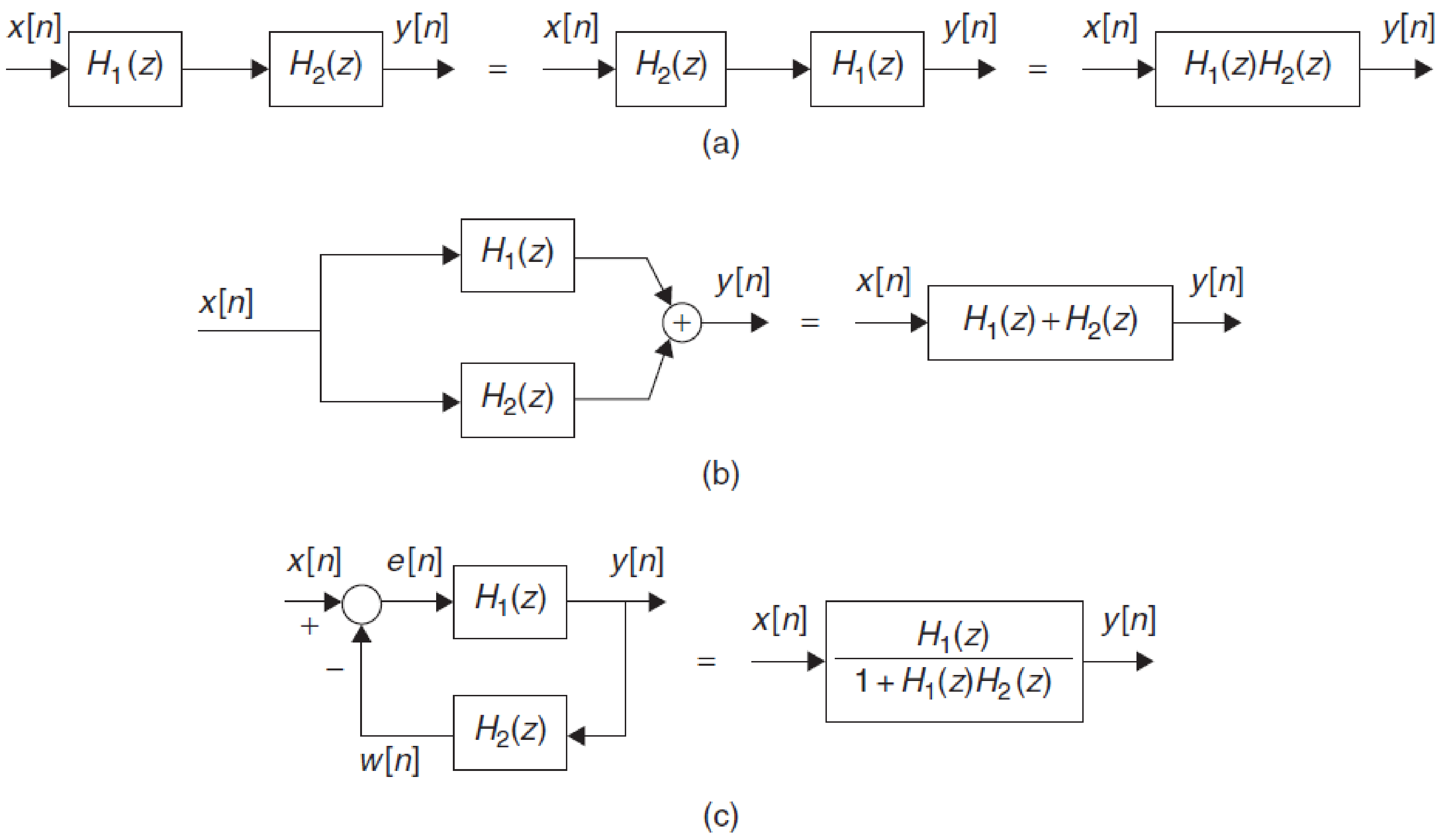
- Hence,

$$y[n] = \sum_{k=0}^{N-1} h[k] x[n-k] \qquad Y(z) = H(z)X(z)$$

Convolution Sum Length

- The length of the convolution sum of two sequences of lengths M and N is M+N-1
- If one of the sequences is of infinite length, the length of the convolution is infinite
- Thus, for an *Infinite Impulse Response (IIR)* or recursive filters the output is always of infinite length for any input signal, given that the impulse response of these filters is of infinite length

Interconnecting Discrete-Time Systems



Initial and Final Value Properties

Initial value: $x[0] = \lim_{z \to \infty} X(z)$

Final value: $\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1)X(z)$

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} \left(x[0] + \sum_{n \geq 1} \frac{x[n]}{z^n} \right) = x[0]$$

$$(z-1)X(z) = \sum_{n=0}^{\infty} x[n]z^{-n+1} - \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$= x[0]z + \sum_{n=0}^{\infty} [x[n+1] - x[n]]z^{-n}$$

$$\lim_{z \to 1} (z-1)X(z) = x[0] + \sum_{n=0}^{\infty} (x[n+1] - x[n])$$

$$= x[0] + (x[1] - x[0]) + (x[2] - x[1]) + (x[3] - x[2]) \cdots$$

$$= \lim_{n \to \infty} x[n]$$

Use to check on your Z-Transform
or Inverse Z-Transform

Table 9.1 One-Sided Z-Transforms

	Function of Time	Function of z, ROC
1.	$\delta[n]$	1, whole z-plane
2.	$u[n]$	$\frac{1}{1-z^{-1}}$, $\lvert z\rvert > 1$
3.	$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$, $\lvert z\rvert > 1$
4.	$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$, $\lvert z\rvert > 1$
5.	$\alpha^n u[n]$, $\lvert\alpha\rvert < 1$	$\frac{1}{1-\alpha z^{-1}}$, $\lvert z\rvert > \lvert\alpha\rvert$
6.	$n\alpha^n u[n]$, $\lvert\alpha\rvert < 1$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$, $\lvert z\rvert > \lvert\alpha\rvert$
7.	$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > 1$
8.	$\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > 1$
9.	$\alpha^n\cos(\omega_0 n)u[n]$, $\lvert\alpha\rvert < 1$	$\frac{1-\alpha\cos(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > 1$
10.	$\alpha^n\sin(\omega_0 n)u[n]$, $\lvert\alpha\rvert < 1$	$\frac{\alpha\sin(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > \lvert\alpha\rvert$

Table 9.2 Basic Properties of One-Sided Z-Transform

Causal signals and constants	$\alpha x[n], \beta y[n]$	$\alpha X(z), \beta Y(z)$
Linearity	$\alpha x[n] + \beta y[n]$	$\alpha X(z) + \beta Y(z)$
Convolution sum	$(x * y)[n] = \sum_k x[n]y[n-k]$	$X(z)Y(z)$
Time shifting—causal	$x[n-N]$ N integer	$z^{-N}X(z)$
Time shifting—noncausal	$x[n-N]$	$z^{-N}X(z) + x[-1]z^{-N+1}$
	$x[n]$ noncausal, N integer	$+ x[-2]z^{-N+2} + \cdots + x[-N]$
Time reversal	$x[-n]$	$X(z^{-1})$
Multiplication by n	$n\,x[n]$	$-z\dfrac{dX(z)}{dz}$
Multiplication by n^2	$n^2\,x[n]$	$z^2\dfrac{d^2X(z)}{dz^2} + z\dfrac{dX(z)}{dz}$
Finite difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z) - x[-1]$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\dfrac{X(z)}{1 - z^{-1}}$
Initial value	$x[0]$	$\lim_{z\to\infty} X(z)$
Final value	$\lim_{n\to\infty} x[n]$	$\lim_{z\to 1}(z-1)X(z)$

Inverse Z-Transform (One-Sided)

- Method #1: If the Z-transform is given as a finite-order polynomial, the inverse can be found by inspection

$$X(z) = \sum_{n=0}^{N} x[n] z^{-n}$$

$$= x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots + x[N]z^{-N}$$

- Example:

$$X(z) = 1 + 2z^{-10} + 3z^{-20} \quad \Rightarrow \quad x[n] = \delta[n] + 2\delta[n-10] + 3\delta[n-20]$$

Inverse Z-Transform (One-Sided)

- Method #2: Partial Fraction Expansion for rational functions given as $X(z) = B(z)/A(z)$
- To find the inverse we simply divide the polynomial $B(z)$ by (z) to obtain a possible infinite-order polynomial in negative powers of z^{-1}
  - Coefficients of this polynomial are the inverse values
- Disadvantage: it does not provide a closed-form solution
  - Useful when interested to get a few initial values of $x[n]$
- Example:

$$X(z) = \frac{1}{1 + 2z^{-2}} = 1 + (-2z^{-2})^1 + (-2z^{-2})^2 + (-2z^{-2})^3 + \cdots$$

$$x[0] = 1$$
$$x[1] = 0$$
$$x[2] = -2$$
$$x[3] = 0$$
$$x[4] = (-2)^2$$

Inverse Z-Transform (One-Sided)

- Method #3: Partial Fraction Expansion
  - Similar to Laplace transform

$$X(z) = \frac{N(z)}{D(z)}$$

- Example: $X(z) = \dfrac{1 + z^{-1}}{(1 + 0.5z^{-1})(1 - 0.5z^{-1})}$

$$X(z) = \frac{1 + z^{-1}}{(1 + 0.5z^{-1})(1 - 0.5z^{-1})}$$

$$= \frac{A}{1 + 0.5z^{-1}} + \frac{B}{1 - 0.5z^{-1}}$$

$$X(z) = \frac{-0.5z}{z + 0.5} + \frac{1.5z}{z - 0.5}$$

$$x[n] = [-0.5(-0.5)^n + 1.5(0.5)^n]u[n]$$

Solution of Difference Equations

- Use the shifting in time property of the Z-transform in the solution of difference equations with initial conditions
  - Very similar to Laplace transform when solving differential equations

Time shifting—causal	$x[n-N]$ N integer	$z^{-N}X(z)$
Time shifting—noncausal	$x[n-N]$	$z^{-N}X(z)+x[-1]z^{-N+1}$
	$x[n]$ noncausal, N integer	$+x[-2]z^{-N+2}+\cdots+x[-N]$

$$y[n] = y_{zs}[n] + y_{zi}[n]$$

Solution of Difference Equations: Example 1

- Solve the following difference equation with zero initial conditions and x[n]=u[n]

$$y[n] = y[n-1] - 0.25y[n-2] + x[n] \qquad n \geq 0$$

Solution:

$$Y(z) = \frac{X(z)}{1 - z^{-1} + 0.25z^{-2}}$$

$$= \frac{1}{(1 - z^{-1})(1 - z^{-1} + 0.25z^{-2})} = \frac{z^3}{(z-1)(z^2 - z + 0.25)} \qquad |z| > 1$$

$$\Rightarrow \quad y[n] = Au[n] + [B(0.5)^n + Cn(0.5)^n]\,u[n]$$

Solution of Difference Equations: Example 2

- Fine the complete response for the following difference equation:

$$y[n] + y[n-1] - 4y[n-2] - 4y[n-3] = 3x[n] \qquad n \geq 0$$

$$y[-1] = 1$$

$$y[-2] = y[-3] = 0$$

$$x[n] = u[n]$$

- Solution:

$$Y(z)[1 + z^{-1} - 4z^{-2} - 4z^{-3}] = 3X(z) + [-1 + 4z^{-1} + 4z^{-2}]$$

$$Y(z) = 3\frac{X(z)}{A(z)} + \frac{-1 + 4z^{-1} + 4z^{-2}}{A(z)} \qquad |z| > 2$$

$$A(z) = 1 + z^{-1} - 4z^{-2} - 4z^{-3} = (1 + z^{-1})(1 + 2z^{-1})(1 - 2z^{-1})$$

$$\boxed{y[n] = y_{zs}[n] + y_{zi}[n]}$$

Problem Assignments

- Problems: **9.3**, 9.7, **9.8**, 9.9, 9.10, **9.11**, 9.14, 9.16, 9.17, **9.18**, 9.19
- Partial Solutions available from the student section of the textbook web site