Chapter 2

Chapter 2 (Intermediate Physics)
2008-2009

Web: http://ymk.k-space.org/courses.htm

Exponential Growth

- An exponential growth process is one in which the rate of increase of a quantity is proportional to that quantity
- Example:

Savings account

$$y_t = y_0(1+b)^t$$

TABLE 2.1. Growth of a savings account earning 5% interest compounded annually, when the initial investment is $100.

Year	Amount	Year	Amount	Year	Amount
1	$105.00	10	$162.88	100	$13,150.13
2	110.25	20	265.33	200	1,729,258.09
3	115.76	30	432.19	300	2.27×10^{8}
4	121.55	40	704.00	400	2.99×10^{10}
5	127.63	50	1146.74	500	3.93×10^{12}
6	134.01	60	1867.92	600	5.17×10^{14}
7	140.71	70	3042.64	700	6.80×10^{16}
8	147.75	80	4956.14	800	8.94×10^{18}
9	155.13	90	8073.04	900	1.18×10^{21}

Exponential Growth

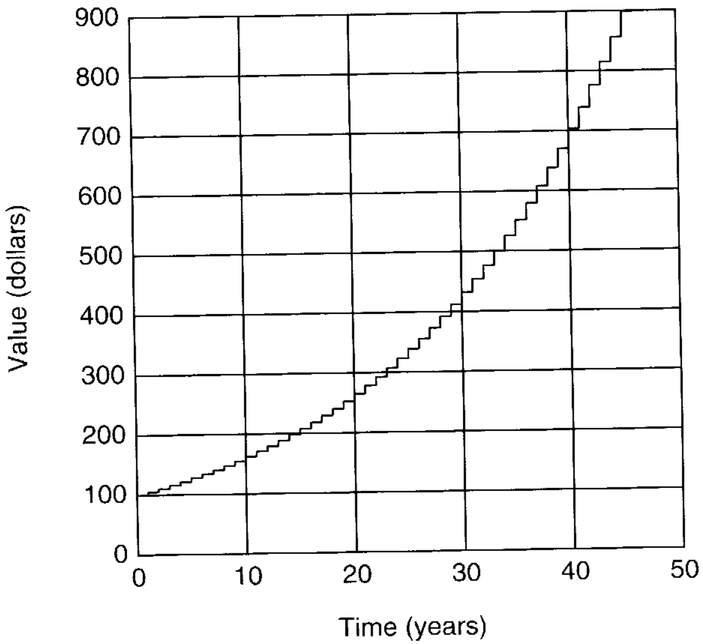

FIGURE 2.1. The amount in a savings account after t years when the amount is compounded annually at 5% interest.

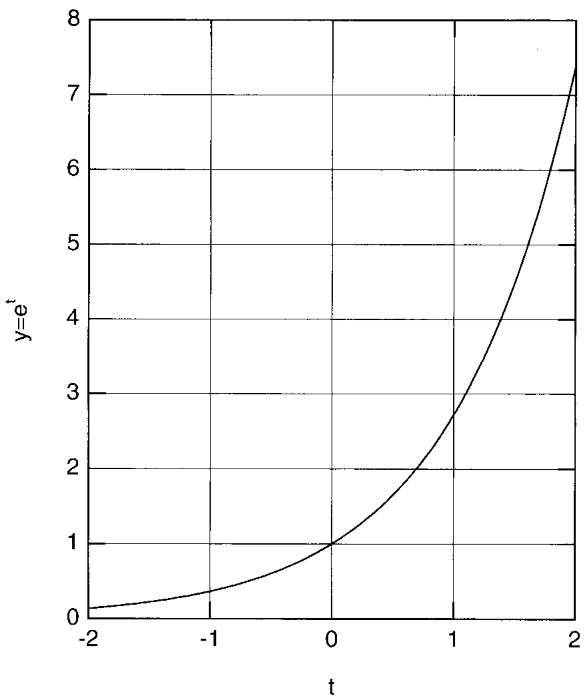

FIGURE 2.2. A graph of the exponential function $y = e^t$

Exponential Growth: Compounding

- N times/year

$$y_t = y_0\left(1+\frac{b}{N}\right)^{Nt}$$

TABLE 2.2. Amount of an initial investment of \$100 at 5% annual interest, with different methods of compounding.

Month	Annual	Semiannual	Quarterly	Monthly	Instant
0	\$100.00	\$100.00	\$100.00	\$100.000	\$100.000
1	100.00	100.00	100.00	100.417	100.418
2	100.00	100.00	100.00	100.835	100.837
3	100.00	100.00	101.25	101.255	101.258
4	100.00	100.00	101.25	101.677	101.681
5	100.00	100.00	101.25	102.101	102.105
6	100.00	102.50	102.52	102.526	102.532
7	100.00	102.50	102.52	102.953	102.960
8	100.00	102.50	102.52	103.382	103.390
9	100.00	102.50	103.80	103.813	103.821
10	100.00	102.50	103.80	104.246	104.255
11	100.00	102.50	103.80	104.680	104.690
12	105.00	105.06	105.09	105.116	105.127

Exponential Growth

- As compounding N $\to\infty$,

$$y_t = \lim_{N\to\infty}\left\{ y_0\left(1+\frac{b}{N}\right)^{Nt}\right\}$$

$$= \lim_{N\to\infty}\left\{ y_0\left[\left(1+\frac{b}{N}\right)^{N}\right]^{t}\right\}$$

$$= y_0 e^{bt}$$

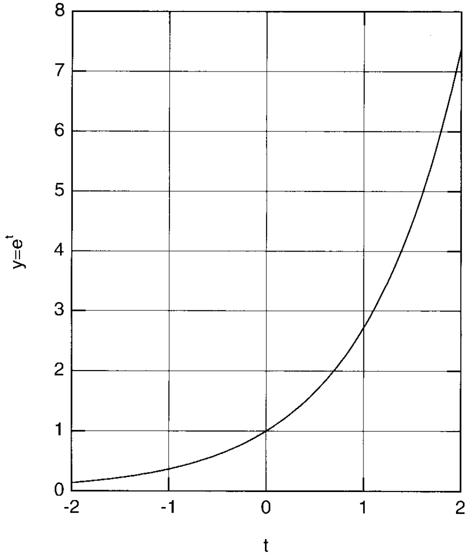

FIGURE 2.2. A graph of the exponential function $y = e^t$

Exponential Growth

- Differential Equation

$$\frac{dy_t}{dt} = \frac{d}{dt}\left\{y_0 e^{bt}\right\} = by_0 e^{bt} = by_t$$

Exponential Decay

- Example: assume b>0

$$y_t = y_0 e^{-bt}$$

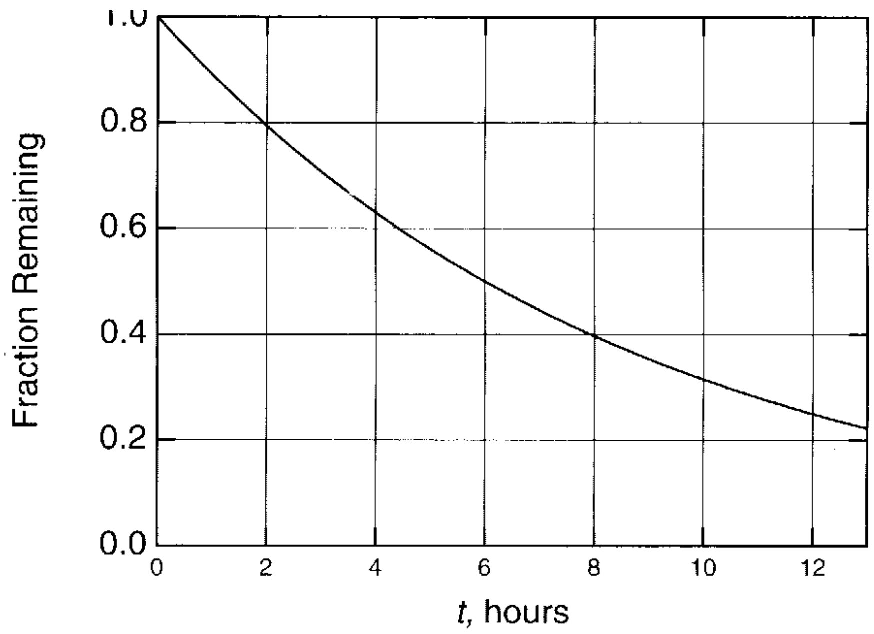

FIGURE 2.3. A plot of the fraction of nuclei of ^{99m}Tc surviving at time t.

Exponential Decay

- Half-Life $T_{½}$: Lebgth of time required for y_t to decrease to ½ its original value

$$y_{T_{1/2}} = 0.5 y_0 \Leftrightarrow e^{-bT_{1/2}} = 0.5$$

$$T_{1/2} \approx \frac{0.693}{b}$$

- Note: Doubling time T_2 is same value

Exponential Decay

- Example: Radioactive decay of ^{99m}Tc
  - Decay rate: b= 0.1155 h^{-1}
  - $T_{½}$ = 0.693/0.1155 = 6 h

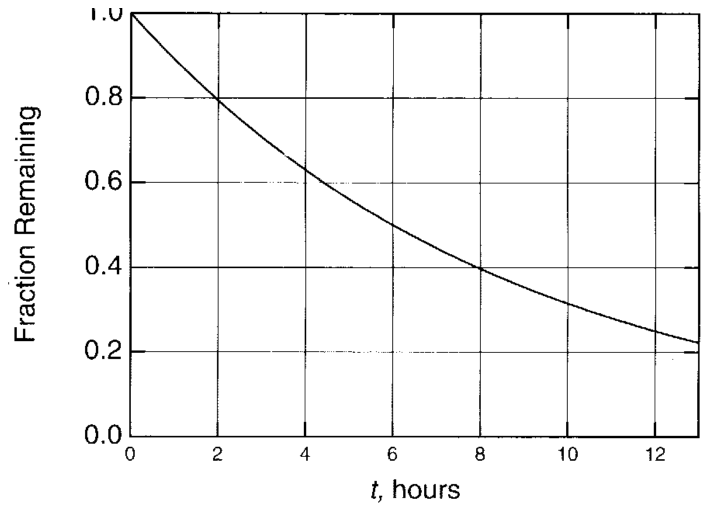

FIGURE 2.3. A plot of the fraction of nuclei of ^{99m}Tc surviving at time t.

Semilog Paper

$$\log y_t = \log y_0 e^{bt} = \log y_0 + bt$$

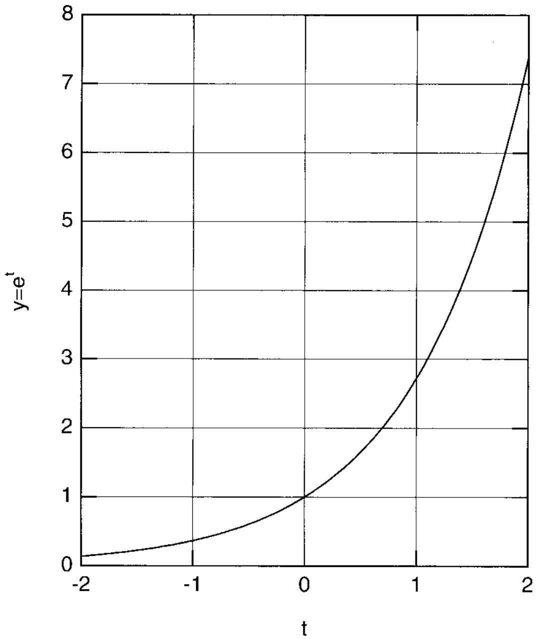

FIGURE 2.2. A graph of the exponential function $y = e^t$

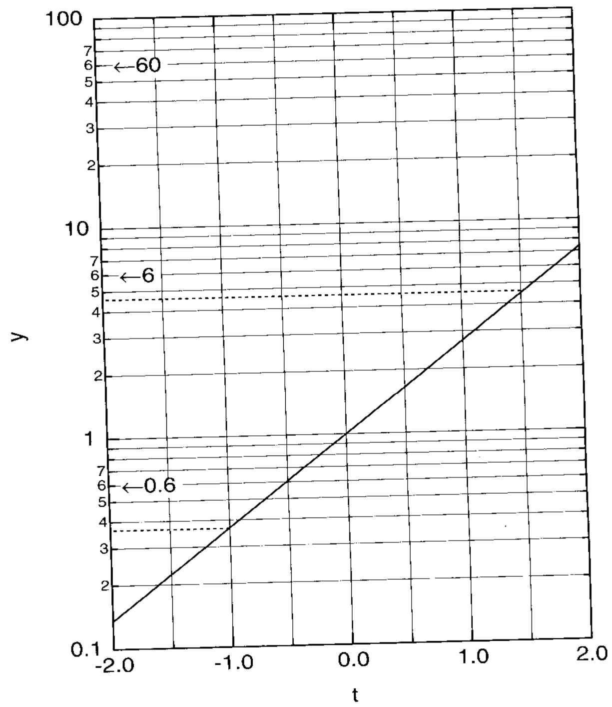

FIGURE 2.4. A plot of the exponential function on semilog paper.

Semilog Paper: Example

$$\log y_t = \log y_0 e^{-bt}$$
$$= \log y_0 - bt$$

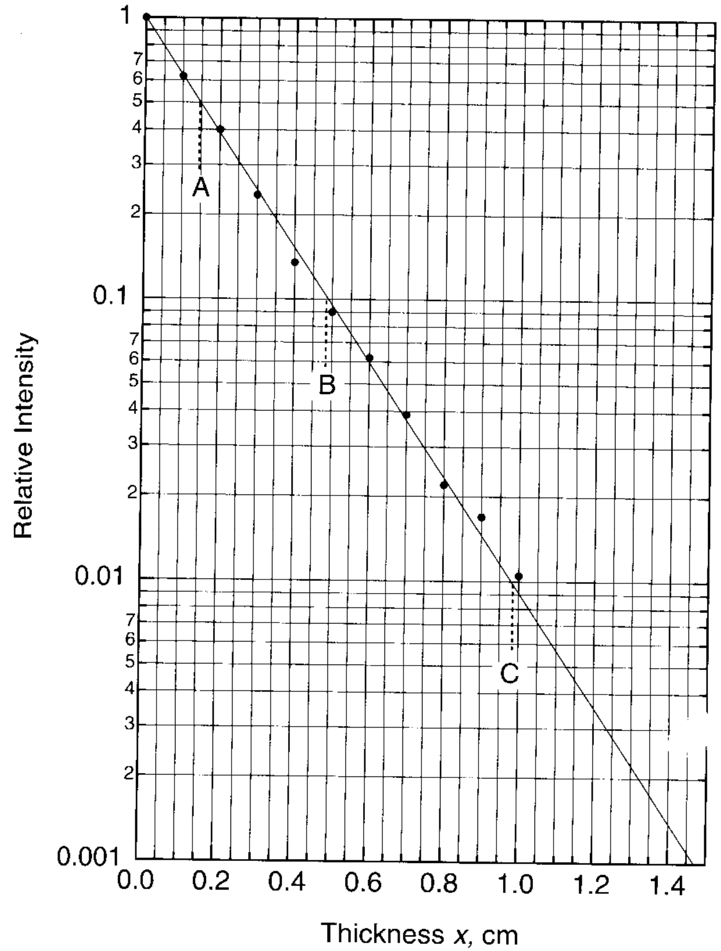

FIGURE 2.5. A semilogarithmic plot of the intensity of light after it has passed through an absorber of thickness x.

Variable Rates

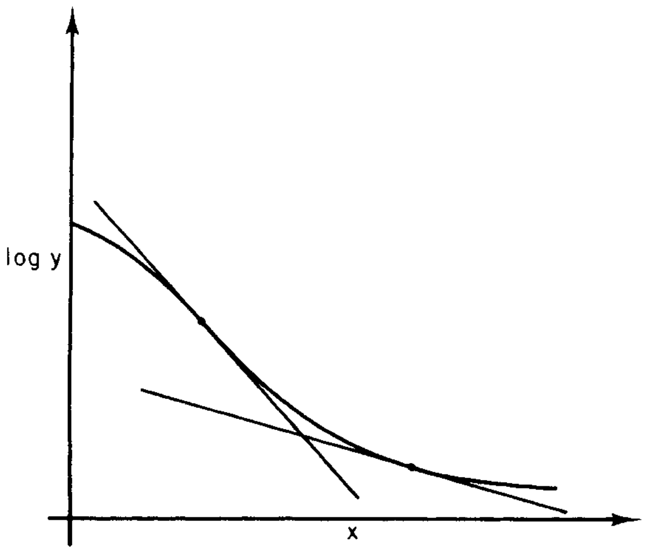

FIGURE 2.6. A semilogarithmic plot of y vs x when the decay rate is not constant. Each tangent line represents the instantaneous decay rate for that value of x.

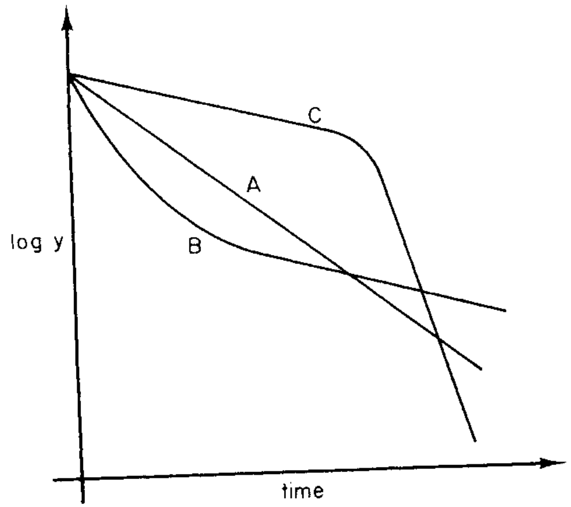

FIGURE 2.7. Semilogarithmic plots of the fraction of a population surviving in three different diseases. The death rates (decay constants) depend on the duration of the disease.

Variable Rates: Example

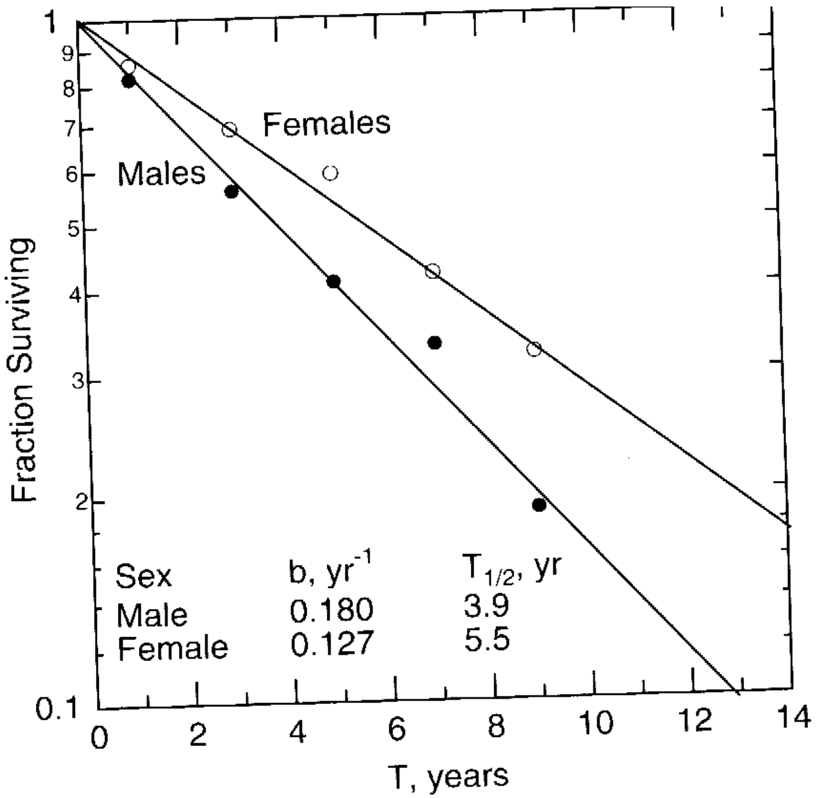

FIGURE 2.8. Survival of patients with congestive heart failure. Data are from McKee *et al.* (1971).

Variable Rates: Example

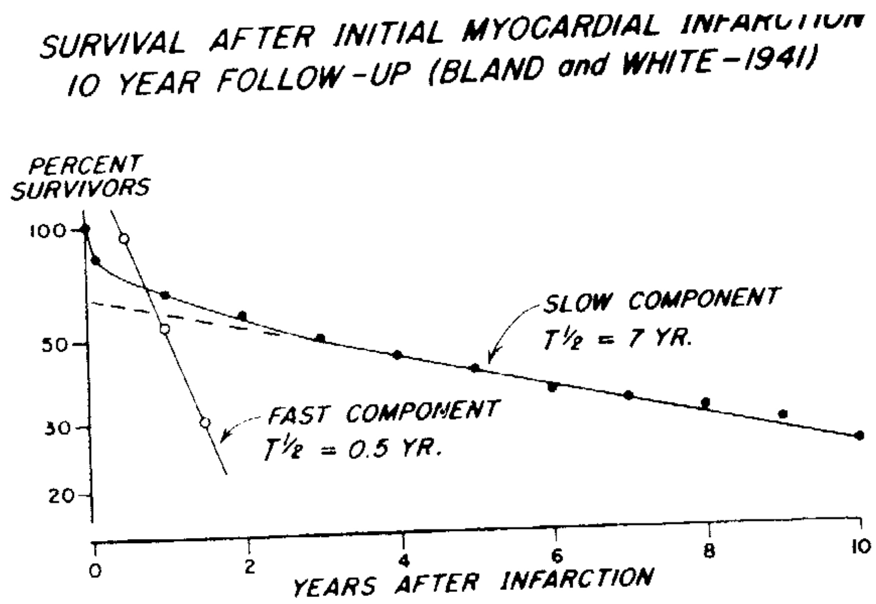

FIGURE 2.9. The fraction of patients surviving after a
ocardial infarction (heart attack) at $t = 0$. The curve lab
"Fast Component" plots 10 times the difference between
survival curve and the extrapolated "Slow Component." F
B. Zumoff, H. Hart, and L. Hellman (1966). Considerat
of mortality in certain chronic diseases. *Ann. Intern.*
64: 595–601. Reproduced by permission of *Annals of Inte*
Medicine. Drawing courtesy of Prof. Zumoff.

Clearance

- Clearance K is defined by,

$$\frac{dy}{dt} = -KC = -K\left(\frac{y}{V}\right) = -\left(\frac{K}{V}\right)y$$

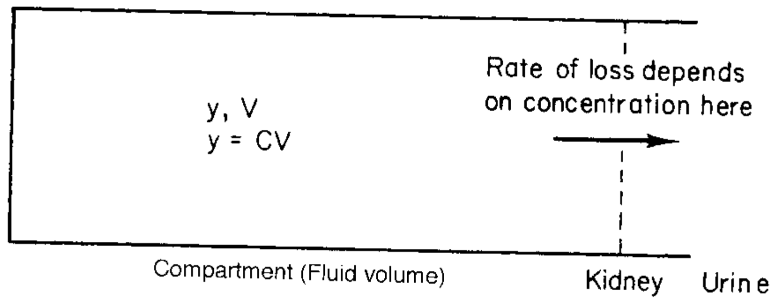

FIGURE 2.10. A case in which the rate of removal of a substance from the a fluid compartment depends on the concentration, not on the total amount of substance in the compartment. Increasing the compartment volume with the same concentration of the substance would not change the rate of removal.

Multiple Decay Paths

- Multiple decay processes

$$\frac{dy}{dt} = -b_1 y - b_2 y - b_3 y - \cdots = -(b_1 + b_2 + b_3 + \cdots) y = -by$$

- Half-life

$$\frac{0.693}{T} = \frac{0.693}{T_1} + \frac{0.693}{T_2} + \frac{0.693}{T_3} + \cdots \Rightarrow T = (T_1 + T_2 + T_3 + \cdots)$$

Decay Plus Input at a Constant Rate

$$\frac{dy}{dt} = a - by \Rightarrow y = \frac{a}{b}(1 - e^{-bt})$$

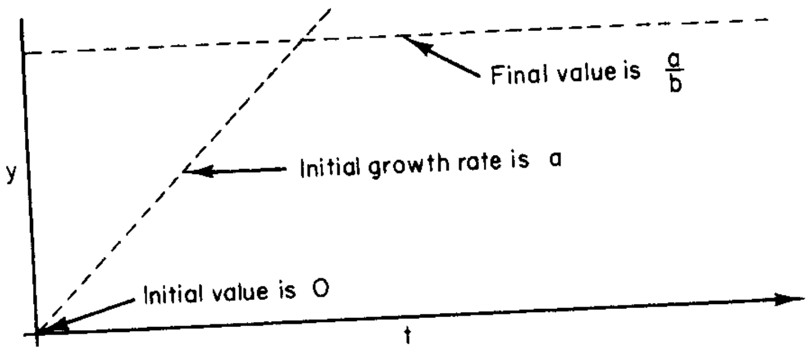

FIGURE 2.11. Sketch of the initial slope a and final value a/b of y when $y(0) = 0$.

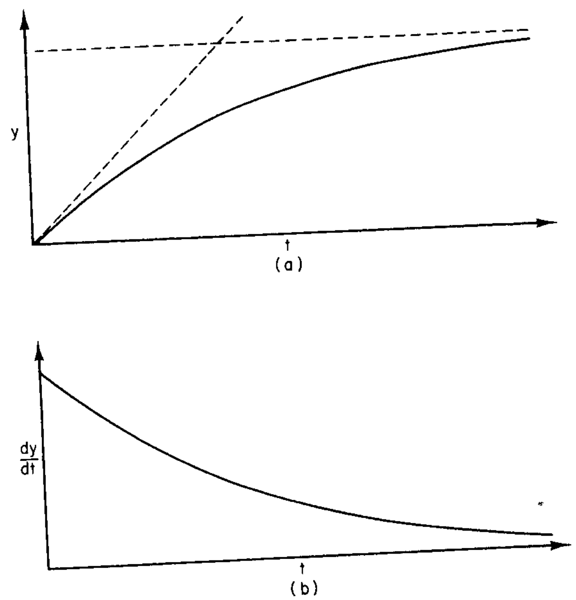

FIGURE 2.12. (a) Plot of $y(t)$. (b) Plot of dy/dt.

Decay with Multiple Half-Lives: Fitting Exponentials

$$y = y_1 + y_2$$

$$= A_1 e^{-b_1 t} + A_2 e^{-b_2 t}$$

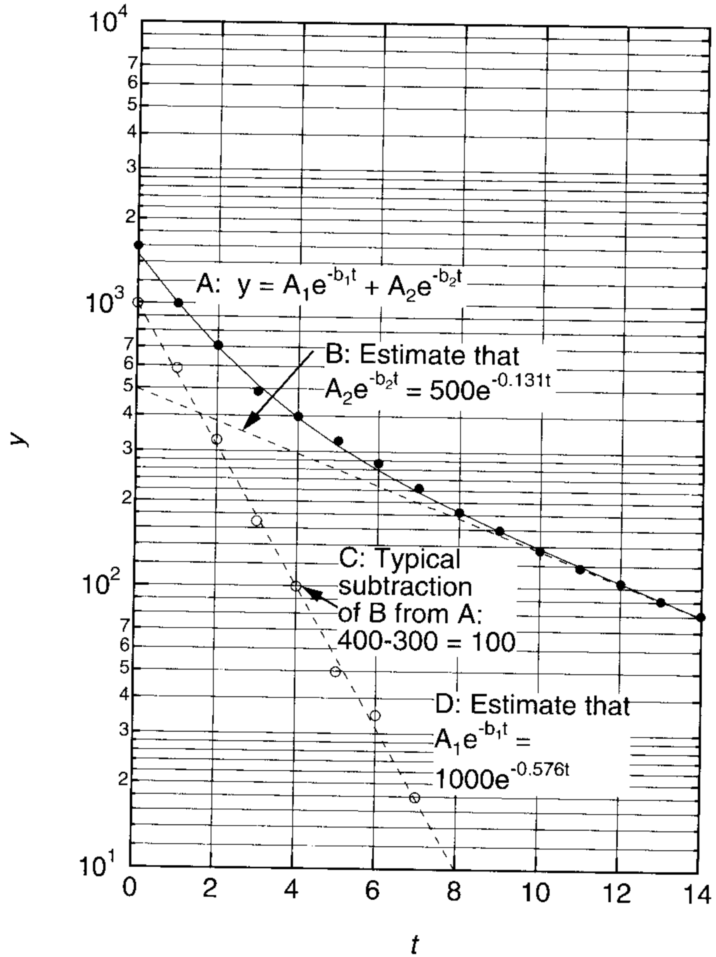

FIGURE 2.13. Fitting a curve with two exponentials.

Log-Log Plots

$$y = Bx^n$$

$$\log y = \log B + n \log x$$

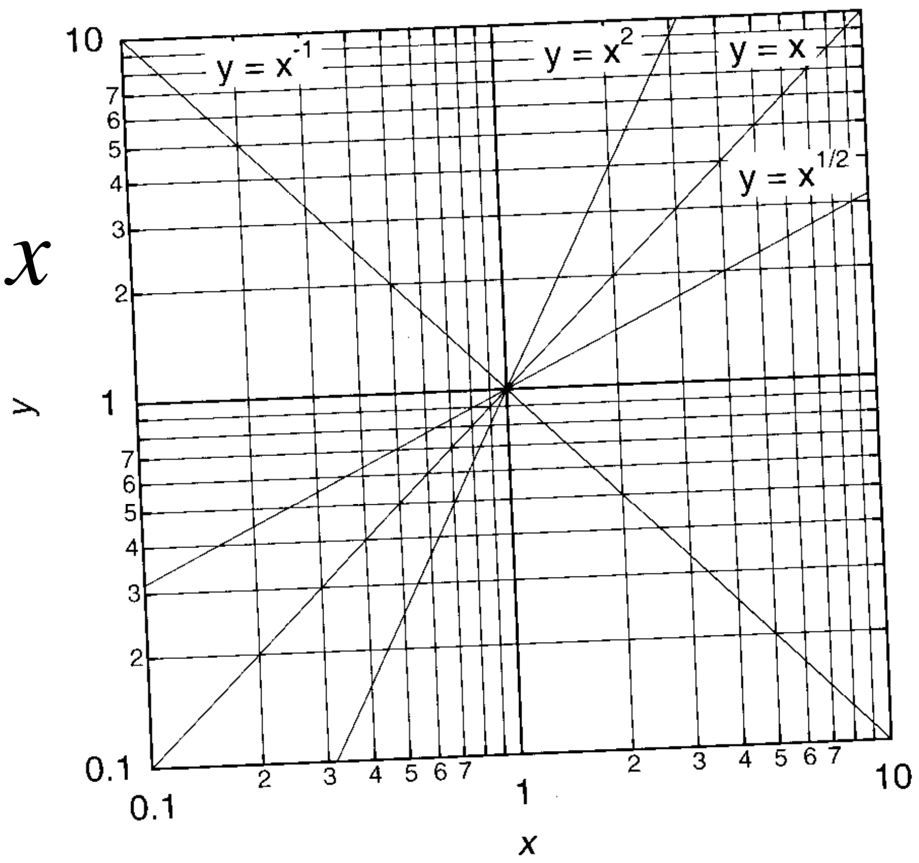

FIGURE 2.15. Log-log plots of $y = x^n$ for different values of n. When $x = 1$, $y = 1$ in every case.

Log-Log Plots

- Scaling
- Nonzero intercep

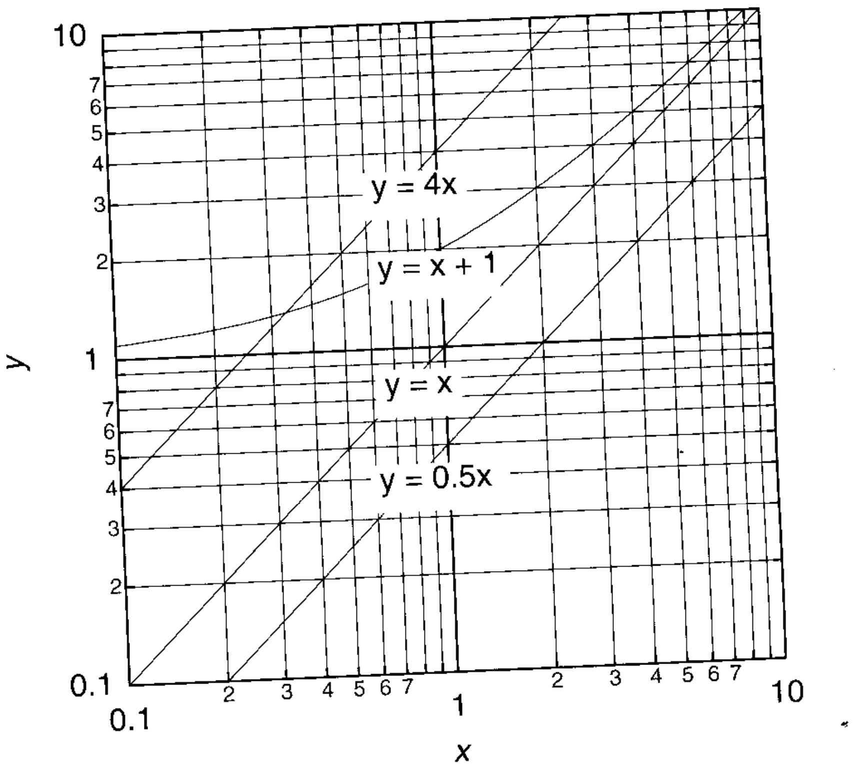

FIGURE 2.16. Log–log plots of $y = Bx$, showing how the curves shift on the paper as B changes. Since $n = 1$ for all the curves, they all have the same slope. There is also a plot of $y = x + 1$, to show that a polynomial does not plot as a straight line.

Example: Food Consumption

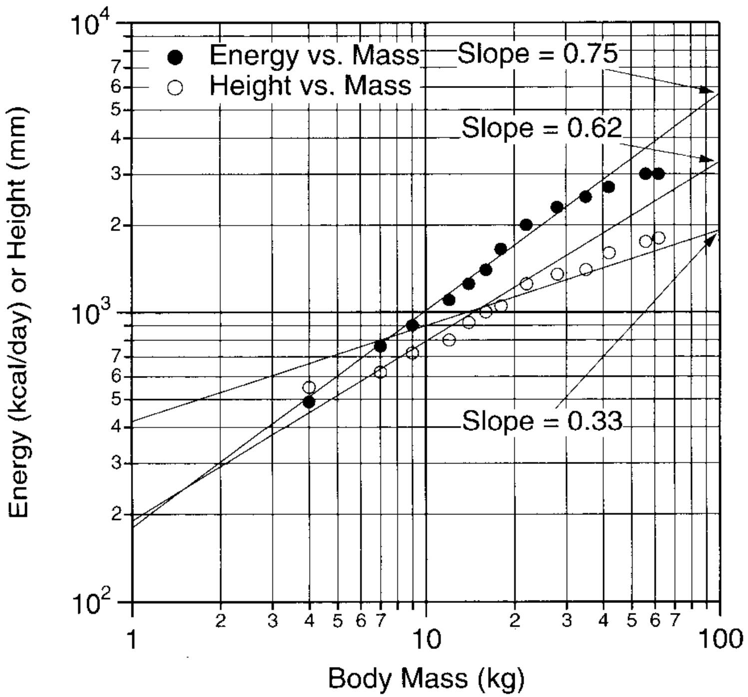

FIGURE 2.17. Plot of daily food requirement F and height H vs mass M for growing children. Data are from Kempe *et al.* (1970), p. 90.

Example: Basal Metabolic Rate

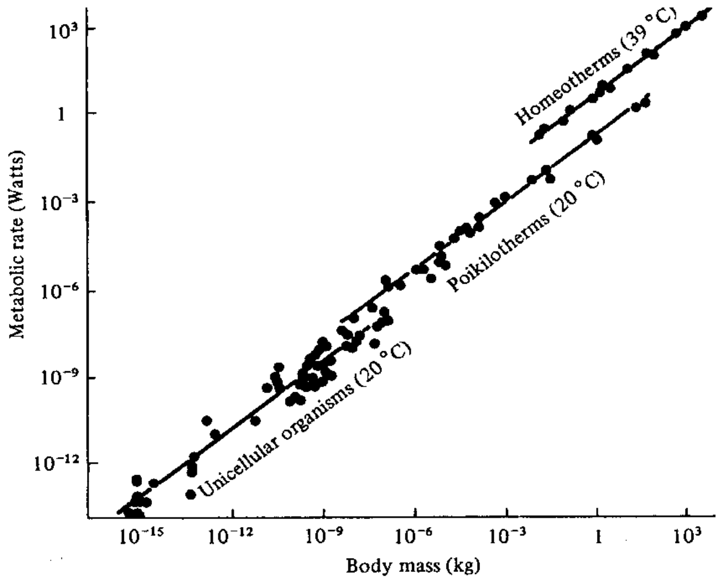

FIGURE 2.18. Plot of resting metabolic rate vs. body mass for many different organisms. Graph is from R. H. Peters (1983). *The Ecological Implications of Body Size*. Cambridge, Cambridge University Press. Modified from A. M. Hemmingsen (1960). Energy metabolism as related to body size and respiratory surfaces, and its evolution. *Reports of the Steno Memorial Hospital and Nordisk Insulin Laboratorium*. **9** (Part II): 6–110. Used with permission.

Problem Assignment

- Posted on class web site
- Solution manual is available from the textbook's web site