CAIRO UNIVERSITY

Medical Image Reconstruction Term II – 2012

Topic 2: Fourier Optics Basics

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Fourier Transform

- Forward transform (*Analysis*)

$$\mathcal{F}\{g\} = \iint_{-\infty}^{\infty} g(x, y) \exp[-j2\pi(f_X x + f_Y y)]\, dx\, dy.$$

- Inverse transform (*Synthesis*)

$$\mathcal{F}^{-1}\{G\} = \iint_{-\infty}^{\infty} G(f_X, f_Y) \exp[j2\pi(f_X x + f_Y y)]\, df_X\, df_Y.$$

Existence of Fourier Transform

- Sufficient (not necessary) conditions:
  - g absolute-integrable
  - g has finite discontinuities/max/min
  - g has no infinite discontinuities
- Bracewell: "physical possibility is a valid sufficient condition for the existence of a transform"
  - Example: dirac-delta function

Fourier Transform as a Decomposition: 1D Case

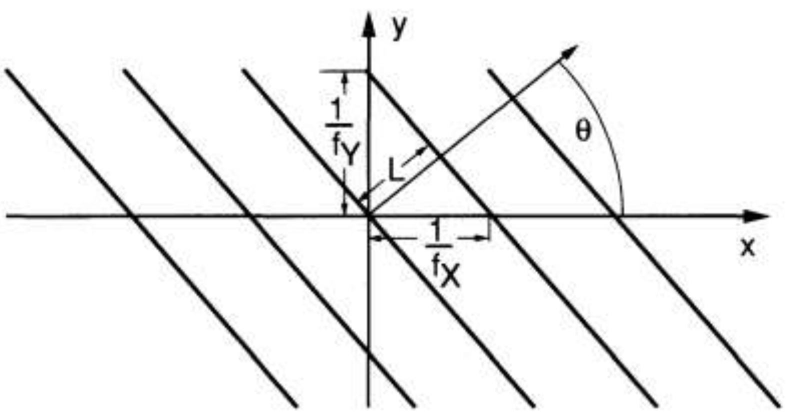
- Linearity enables decomposition into sum of elementary functions
- Fourier analysis is an example of such decomposition:

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi f t)\,df$$

  - Physical realization: pure harmonics
  - Weighting functions: complex G values

Fourier Transform as a Decomposition: 2D Case

- Elementary functions: 2D harmonics
- Frequency "pair"
  - Physical realization: plane waves
  - Spatial period: distance between zero phase lines (wavefronts)



$$L = \frac{1}{\sqrt{f_X^2 + f_Y^2}}.$$

Fourier Transform Theorems

- Linearity theorem

$$\mathcal{F}\{\alpha g + \beta h\} = \alpha \mathcal{F}\{g\} + \beta \mathcal{F}\{h\}$$

- Similarity theorem

$$\mathcal{F}\{g(ax, by)\} = \frac{1}{|ab|} G\left(\frac{f_X}{a}, \frac{f_Y}{b}\right)$$

Fourier Transform Theorems

- Shift theorem

$$\mathcal{F}\{g(x-a, y-b)\} = G(f_X, f_Y)\exp[-j2\pi(f_X a + f_Y b)]$$

- Parseval's theorem

$$\iint_{-\infty}^{\infty} |g(x, y)|^2\, dx\, dy = \iint_{-\infty}^{\infty} |G(f_X, f_Y)|^2\, df_X\, df_Y$$

Fourier Transform Theorems

- Convolution theorem

$$\mathcal{F}\left\{\iint\limits_{-\infty}^{\infty} g(\xi,\eta)\,h(x-\xi,y-\eta)\,d\xi\,d\eta\right\} = G(f_X,f_Y)\,H(f_X,f_Y)$$

- Autocorrelation theorem

$$\mathcal{F}\left\{\iint\limits_{-\infty}^{\infty} g(\xi,\eta)\,g^*(\xi-x,\eta-y)\,d\xi\,d\eta\right\} = |G(f_X,f_Y)|^2$$

$$\mathcal{F}\{|g(x,y)|^2\} = \iint\limits_{-\infty}^{\infty} G(\xi,\eta)\,G^*(\xi-f_X,\eta-f_Y)\,d\xi\,d\eta$$

Fourier Transform Theorems

- Fourier integral theorem

$$\mathcal{F}\mathcal{F}^{-1}\{g(x, y)\} = \mathcal{F}^{-1}\mathcal{F}\{g(x, y)\} = g(x, y)$$

Separable Functions

- Rectangular coordinates

$$g(x, y) = g_X(x)\, g_Y(y)$$

- Polar coordinates

$$g(r, \theta) = g_R(r)\, g_\Theta(\theta)$$

Fourier Analysis of Separable Functions

- Rectangular
  - Calculation of 2D transform in terms of two 1D transforms

$$\mathcal{F}\{g(x, y)\} = \iint_{-\infty}^{\infty} g(x, y) \exp[-j2\pi(f_X x + f_Y y)]\, dx\, dy$$

$$= \int_{-\infty}^{\infty} g_X(x) \exp[-j2\pi f_X x]\, dx \int_{-\infty}^{\infty} g_Y(y) \exp[-j2\pi f_Y y]\, dy$$

$$= \mathcal{F}_X\{g_X\}\mathcal{F}_Y\{g_Y\}.$$

Fourier Analysis of Separable Functions

- Polar
  - Not as simple
  - Useful cases: circularly symmetric functions

$$g(r, \theta) = g_R(r)$$

  - Fourier-Bessel transform or Hankel transform of 0th order

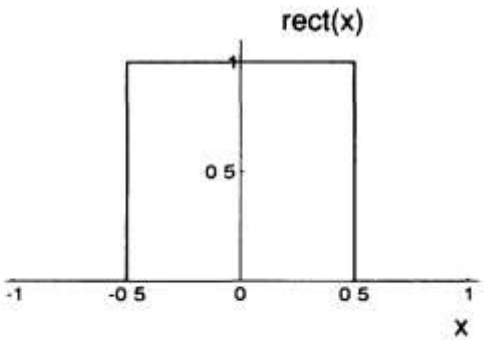
$$G_o(\rho, \phi) = G_o(\rho) = 2\pi \int_0^\infty r g_R(r) J_0(2\pi r \rho)\, dr$$

$$g_R(r) = 2\pi \int_0^\infty \rho G_o(\rho) J_0(2\pi r \rho)\, d\rho$$

Useful Functions

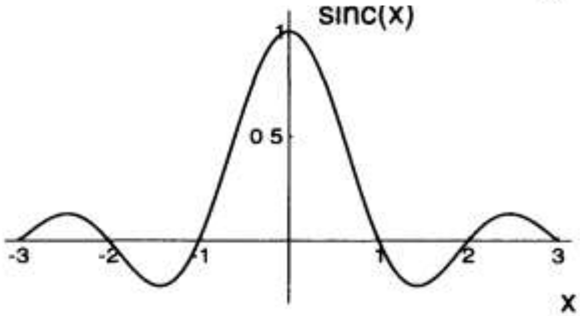
Rectangle function

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



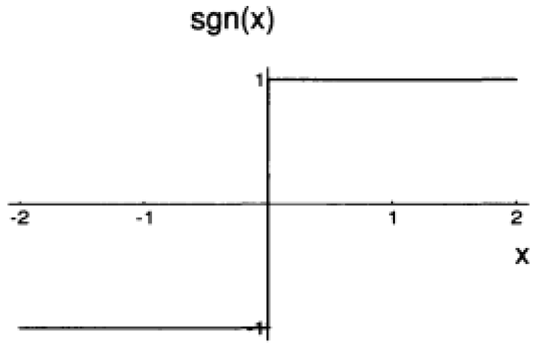
Sinc function

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



Signum function

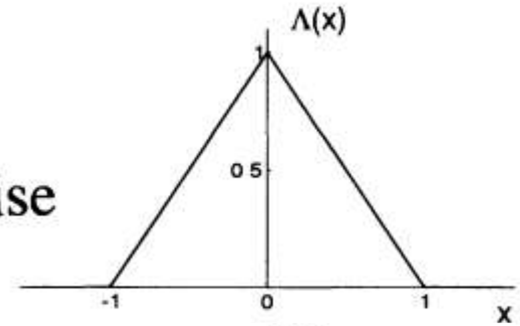
$$\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$



Useful Functions

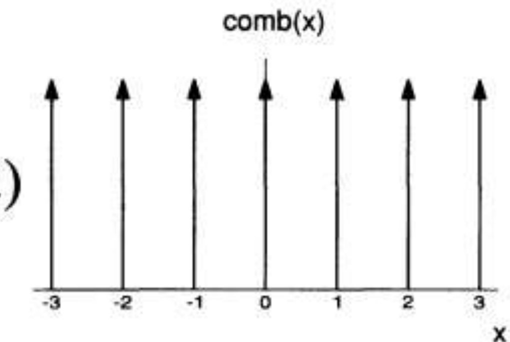
Triangle function

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$



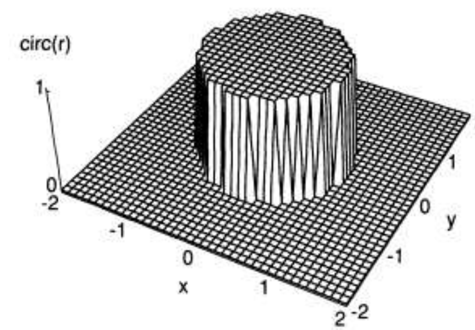
Comb function

$$\operatorname{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$$



Circle function

$$\operatorname{circ}(\sqrt{x^2 + y^2}) = \begin{cases} 1 & \sqrt{x^2 + y^2} < 1 \\ \frac{1}{2} & \sqrt{x^2 + y^2} = 1 \\ 0 & \text{otherwise.} \end{cases}$$



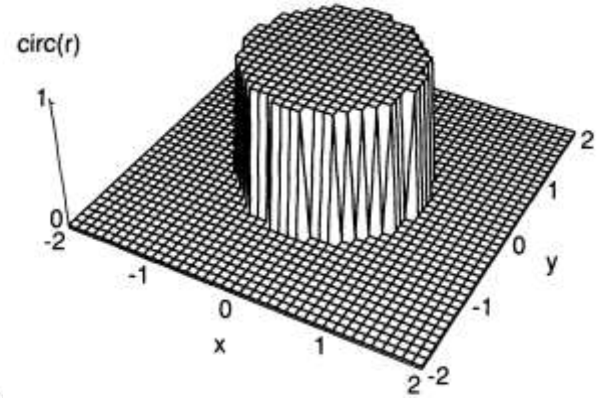
Fourier Transform Pairs

Transform pairs for some functions separable in rectangular coordinates.

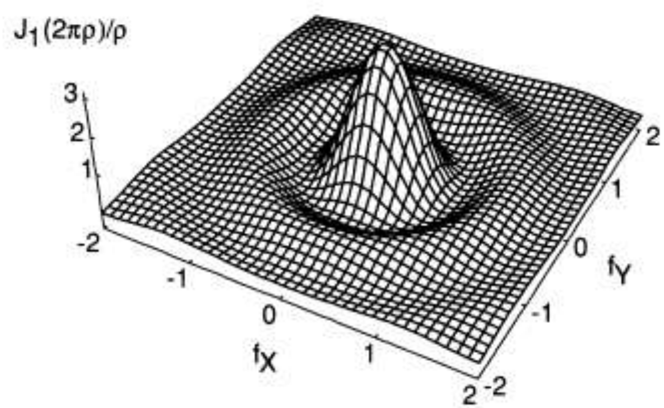
Function	Transform
$\exp[-\pi(a^2x^2 + b^2y^2)]$	$\frac{1}{\lvert ab\rvert} \exp\left[-\pi\left(\frac{f_X^2}{a^2} + \frac{f_Y^2}{b^2}\right)\right]$
$\mathrm{rect}(ax)\,\mathrm{rect}(by)$	$\frac{1}{\lvert ab\rvert} \mathrm{sinc}(f_X/a)\,\mathrm{sinc}(f_Y/b)$
$\Lambda(ax)\,\Lambda(by)$	$\frac{1}{\lvert ab\rvert} \mathrm{sinc}^2(f_X/a)\,\mathrm{sinc}^2(f_Y/b)$
$\delta(ax, by)$	$\frac{1}{\lvert ab\rvert}$
$\exp[j\pi(ax + by)]$	$\delta(f_X - a/2, f_Y - b/2)$
$\mathrm{sgn}(ax)\,\mathrm{sgn}(by)$	$\frac{ab}{\lvert ab\rvert} \frac{1}{j\pi f_X} \frac{1}{j\pi f_Y}$
$\mathrm{comb}(ax)\,\mathrm{comb}(by)$	$\frac{1}{\lvert ab\rvert} \mathrm{comb}(f_X/a)\,\mathrm{comb}(f_Y/b)$
$\exp[j\pi(a^2x^2 + b^2y^2)]$	$\frac{j}{\lvert ab\rvert} \exp\left[-j\pi\left(\frac{f_X^2}{a^2} + \frac{f_Y^2}{b^2}\right)\right]$
$\exp[-(a\lvert x\rvert + b\lvert y\rvert)]$	$\frac{1}{\lvert ab\rvert} \frac{2}{1 + (2\pi f_X/a)^2} \frac{2}{1 + (2\pi f_Y/b)^2}$

Fourier-Bessel Example Pair

$$\mathrm{circ}(r) = \begin{cases} 1 & r < 1 \\ \frac{1}{2} & r = 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\mathcal{B}\{\mathrm{circ}(r)\} = \frac{1}{2\pi\rho^2}\int_0^{2\pi\rho} r' J_0(r')\, dr' = \frac{J_1(2\pi\rho)}{\rho}$$



Local Spatial Frequency

- General function

$$g(x, y) = a(x, y) \exp[j\phi(x, y)]$$

- Local spatial frequency pair defined as:

$$f_{lX} = \frac{1}{2\pi} \frac{\partial}{\partial x} \phi(x, y) \qquad f_{lY} = \frac{1}{2\pi} \frac{\partial}{\partial y} \phi(x, y)$$

- Example:

$$g(x, y) = \exp[j2\pi(f_X x + f_Y y)]$$

$$f_{lX} = \frac{1}{2\pi} \frac{\partial}{\partial x} [2\pi(f_X x + f_Y y)] = f_X$$

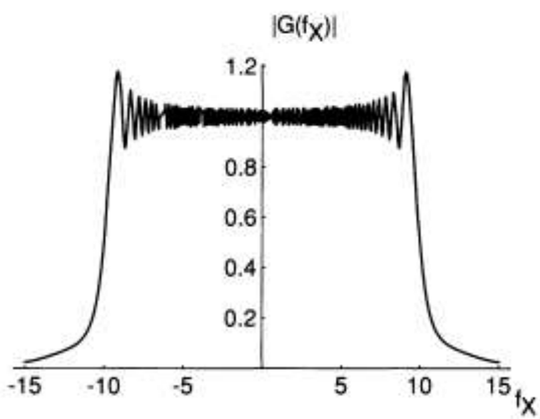
$$f_{lY} = \frac{1}{2\pi} \frac{\partial}{\partial y} [2\pi(f_X x + f_Y y)] = f_Y.$$

Local Spatial Frequency

- Example: finite chirp
  - Local frequencies = 0 when magnitude=0

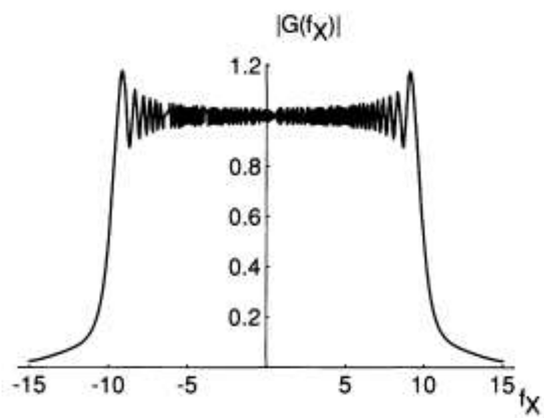
$$g(x, y) = \exp[j\pi\beta(x^2 + y^2)]\,\mathrm{rect}\left(\frac{x}{2L_X}\right)\mathrm{rect}\left(\frac{y}{2L_Y}\right)$$

$$f_{lX} = \beta x\,\mathrm{rect}\left(\frac{x}{2L_X}\right) \qquad f_{lY} = \beta y\,\mathrm{rect}\left(\frac{y}{2L_Y}\right)$$



Space-Frequency Localization

- Since the local spatial frequencies are bounded to covering a rectangle of dimensions $2L_x$ x $2L_y$, we conclude that the Fourier spectrum also limited to same rectangular region.
  - In fact this is approximately true, but not exactly so.



The spectrum of the finite chirp function, $L_X = 10, \beta = 1$.

Linear Systems

- A convenient representation of a system is a mathematical operator S{ }, which we imagine to operate on input functions to produce output functions:

$$g_2(x_2, y_2) = \mathcal{S}\{g_1(x_1, y_1)\}$$

- Linear systems satisfy superposition

$$\mathcal{S}\{ap(x_1, y_1) + bq(x_1, y_1)\} = a\mathcal{S}\{p(x_1, y_1)\} + b\mathcal{S}\{q(x_1, y_1)\}$$

Linear Systems: Impulse Response

$$g_1(x_1, y_1) = \iint\limits_{-\infty}^{\infty} g_1(\xi, \eta)\, \delta(x_1 - \xi, y_1 - \eta)\, d\xi\, d\eta$$

$$g_2(x_2, y_2) = \mathcal{S}\left\{ \iint\limits_{-\infty}^{\infty} g_1(\xi, \eta)\, \delta(x_1 - \xi, y_1 - \eta)\, d\xi\, d\eta \right\}$$

$$g_2(x_2, y_2) = \iint\limits_{-\infty}^{\infty} g_1(\xi, \eta)\, \mathcal{S}\{\delta(x_1 - \xi, y_1 - \eta)\}\, d\xi\, d\eta$$

Define: $h(x_2, y_2; \xi, \eta) = \mathcal{S}\{\delta(x_1 - \xi, y_1 - \eta)\}$. Impulse response

Then, $g_2(x_2, y_2) = \iint\limits_{-\infty}^{\infty} g_1(\xi, \eta)\, h(x_2, y_2; \xi, \eta)\, d\xi\, d\eta$

Spatial Invariance: Transfer Function

- A linear imaging system is space-invariant if its impulse response depends only on the x and y distances between the excitation point and the response point such that:

$$h(x_2, y_2; \xi, \eta) = h(x_2 - \xi, y_2 - \eta).$$

$$g_2(x_2, y_2) = \iint_{-\infty}^{\infty} g_1(\xi, \eta)\, h(x_2 - \xi, y_2 - \eta)\, d\xi\, d\eta$$

$$g_2 = g_1 \otimes h \qquad G_2(f_X, f_Y) = H(f_X, f_Y)\, G_1(f_X, f_Y)$$

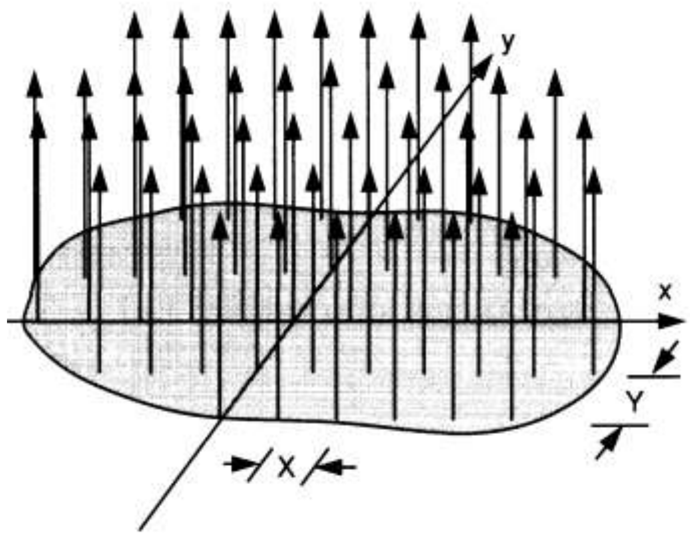
Fourier Transform as Eigen-Decomposition

- Eigenfunction
  - Function that retains its original form up to a multiplicative complex constant after passage through a system
  - Complex-exponential functions are the eigenfunctions of linear, invariant systems.
- Eigenvalue
  - Weighting applied by the system to an eigenfunction input

Whittaker-Shannon Sampling Theorem

- Sampling

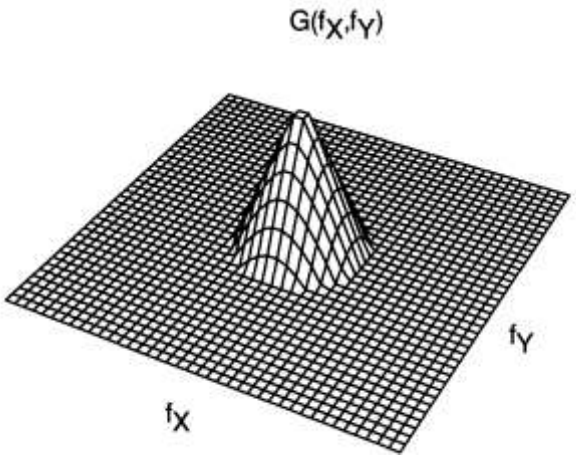
$$g_s(x, y) = \mathrm{comb}\left(\frac{x}{X}\right)\mathrm{comb}\left(\frac{y}{Y}\right)g(x, y).$$

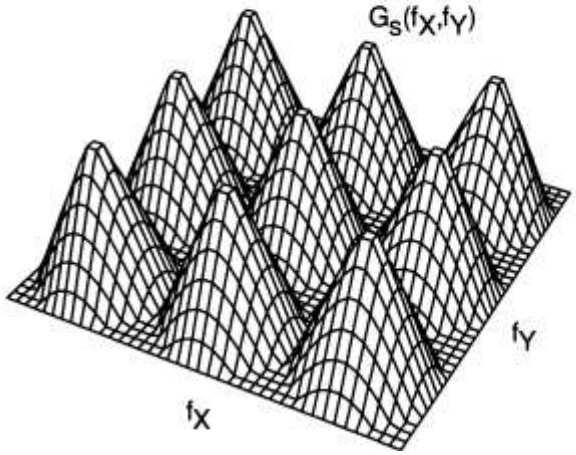


Whittaker-Shannon Sampling Theorem

- Spectrum

$$G_s(f_X, f_Y) = \mathcal{F}\left\{\mathrm{comb}\left(\frac{x}{X}\right)\mathrm{comb}\left(\frac{y}{Y}\right)\right\} \otimes G(f_X, f_Y)$$





$$G_s(f_X, f_Y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G\left(f_X - \frac{n}{X}, f_Y - \frac{m}{Y}\right).$$

Whittaker-Shannon Sampling Theorem

- Exact recovery of a band-limited function can be achieved from an appropriately spaced rectangular array of its sampled values

$$X \leq \frac{1}{2B_X} \qquad \text{and} \qquad Y \leq \frac{1}{2B_Y}.$$

$$g(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} g\left(\frac{n}{2B_X}, \frac{m}{2B_Y}\right) \operatorname{sinc}\left[2B_X\left(x - \frac{n}{2B_X}\right)\right] \operatorname{sinc}\left[2B_Y\left(y - \frac{m}{2B_Y}\right)\right]$$

Space-Bandwidth Product

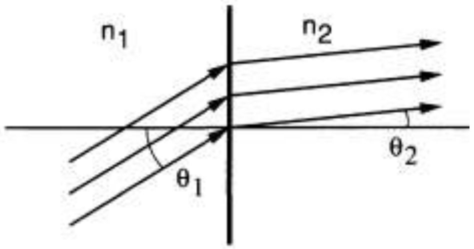
- Measure of complexity
  - Quality of optical system

$$M = 16 L_X L_Y B_X B_Y$$

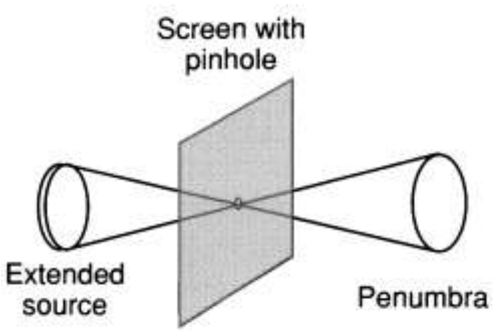
- Has an upper bound for Gaussian functions = $4\pi^2$

Basics of Scalar Diffraction Theory

- Refraction (Snell's law)

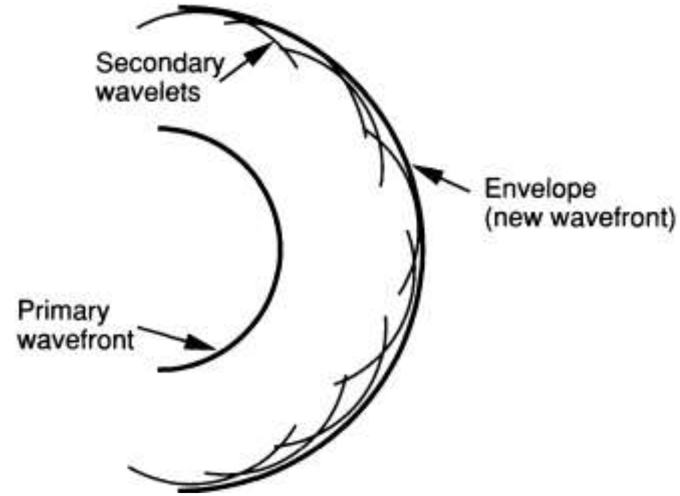


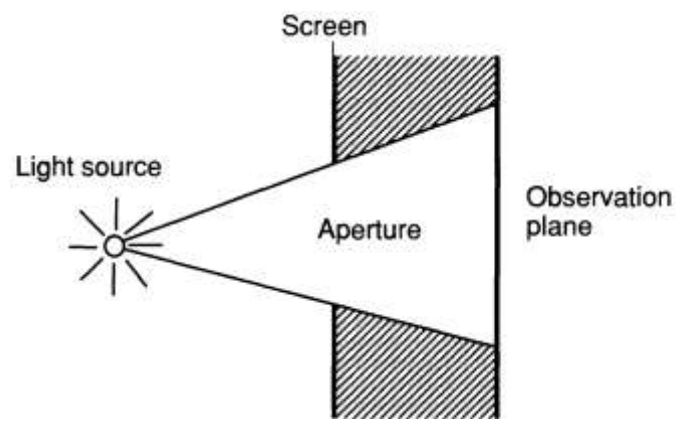
- Diffraction definition (Sommerfeld):
  - Any deviation of light rays from rectilinear paths which cannot be interpreted as reflection or refraction
- Not penumbra effect
  - No bending of rays



Huygens Theory of Light

- Point sources





Major Observations

- Interference (Young)
  - Light can add to darkness
- Wavelet interference (Fresnel)
  - Bright spot at the center of the shadow of an opaque disk (Poisson's spot)
- Maxwell equations

$$\boxed{\begin{aligned} \nabla \times \vec{\mathcal{E}} &= -\mu \frac{\partial \vec{\mathcal{H}}}{\partial t} \\ \nabla \times \vec{\mathcal{H}} &= \epsilon \frac{\partial \vec{\mathcal{E}}}{\partial t} \\ \nabla \cdot \epsilon \vec{\mathcal{E}} &= 0 \\ \nabla \cdot \mu \vec{\mathcal{H}} &= 0. \end{aligned}}$$

- Rayleigh-Sommerfeld diffraction theory

Assumptions about Medium

- Linear
- Isotropic
  - independent of direction of polarization
- Homogeneous
  - Constant permittivity
- Non-dispersive
  - Permittivity is independent of wavelength

Scalar Wave Equation

$$\nabla^2 \vec{\mathcal{E}} - \frac{n^2}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0$$

$$\nabla^2 \vec{\mathcal{H}} - \frac{n^2}{c^2} \frac{\partial^2 \vec{\mathcal{H}}}{\partial t^2} = 0.$$

$$\boxed{\nabla^2 u(P,t) - \frac{n^2}{c^2} \frac{\partial^2 u(P,t)}{\partial t^2} = 0,}$$

Validity of Scalar Theory

- Aperture is large compared to wavelength
  - Comparison to lumped circuit components
- Observations are sufficiently far away from the aperture (many wavelengths)

Helmholtz Equation

- Plug monochromatic wave into scalar wave equation:

$$u(P,t) = Re\{U(P)\exp(-j2\pi\nu t)\},$$

$$\boxed{(\nabla^2 + k^2)U = 0.}$$

- Here wave number

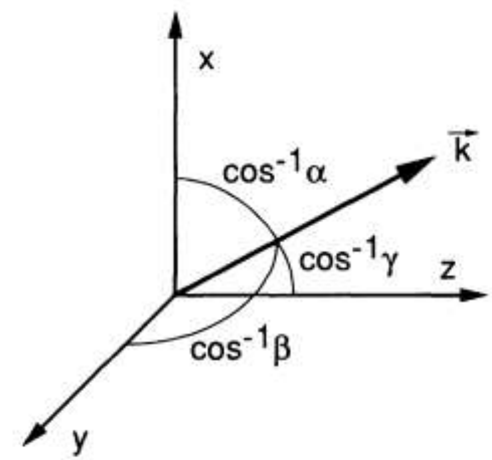
$$k = 2\pi n\frac{\nu}{c} = \frac{2\pi}{\lambda}$$

Plane Waves

- Eigenfunctions of propagation

$$p(x, y, z; t) = \exp[j(\vec{k} \cdot \vec{r} - 2\pi \nu t)]$$

$$\vec{k} = \frac{2\pi}{\lambda}(\alpha\,\hat{x} + \beta\,\hat{y} + \gamma\,\hat{z})$$



- Plane wave at z=0

$$\exp[j2\pi(f_X x + f_Y y)]$$

$$\alpha = \lambda f_X \quad \beta = \lambda f_Y \quad \gamma = \sqrt{1 - (\lambda f_X)^2 - (\lambda f_Y)^2}.$$

Angular Spectrum

- 2D Fourier transform of aperture

$$A(f_X, f_Y; 0) = \iint_{-\infty}^{\infty} U(x, y, 0) \exp[-j2\pi(f_X x + f_Y y)]\, dx\, dy.$$

$$U(x, y, 0) = \iint_{-\infty}^{\infty} A(f_X, f_Y; 0) \exp[j2\pi(f_X x + f_Y y)]\, df_X\, df_Y.$$

- Angular spectrum

$$\boxed{A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0\right) = \iint_{-\infty}^{\infty} U(x, y, 0) \exp\left[-j2\pi\left(\frac{\alpha}{\lambda}x + \frac{\beta}{\lambda}y\right)\right] dx\, dy}$$

Propagation of Angular Spectrum

$$A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; z\right) = A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0\right) \exp(-\mu z)$$

$$\mu = \frac{2\pi}{\lambda}\sqrt{\alpha^2 + \beta^2 - 1}.$$

Propagation as a Linear Spatial Filter

- Free space propagation transfer function

$$\boxed{H(f_X, f_Y) = \begin{cases} \exp\left[j2\pi\frac{z}{\lambda}\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2}\right] & \sqrt{f_X^2+f_Y^2} < \frac{1}{\lambda} \\ 0 & \text{otherwise.} \end{cases}}$$

$$\alpha = \lambda f_X \quad \beta = \lambda f_Y$$

Input Angular Spectrum at z=0 → Propagation Transfer Function → Output Angular Spectrum at z

Fresnel Approximation

- Paraxial (near field) approximation

$$\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2} \approx 1-\frac{(\lambda f_X)^2}{2}-\frac{(\lambda f_Y)^2}{2},$$

$$H(f_X, f_Y) = e^{jkz}\exp\left[-j\pi\lambda z\left(f_X^2+f_Y^2\right)\right].$$

$$h(x, y) = \frac{e^{jkz}}{j\lambda z}\exp\left[\frac{jk}{2z}\left(x^2+y^2\right)\right].$$

Fraunhofer Approximation

- Far field approximation

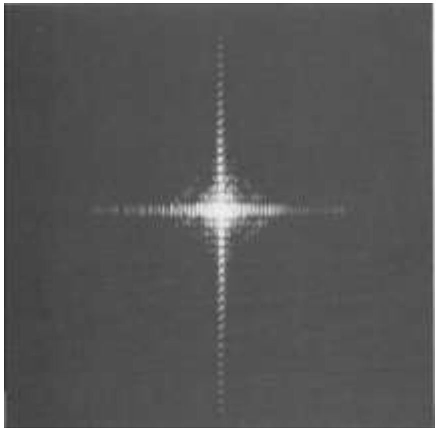
$$z \gg \frac{k(\xi^2 + \eta^2)_{\max}}{2}$$

$$\boxed{U(x, y) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z} \iint_{-\infty}^{\infty} U(\xi, \eta) \exp\left[-j\frac{2\pi}{\lambda z}(x\xi + y\eta)\right] d\xi\, d\eta.}$$

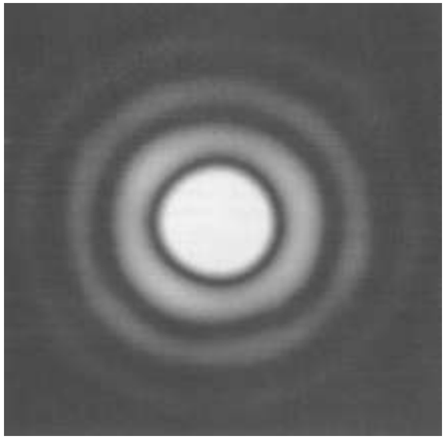
- Field = $\mathcal{F}$\{ Aperture \}

Examples

- Rectangular aperture

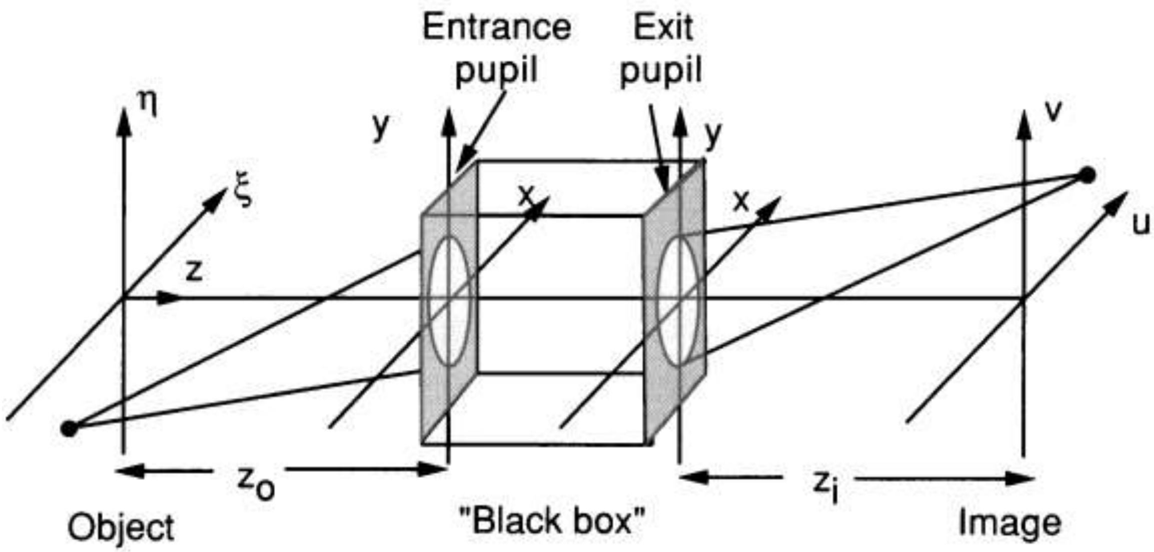


- Circular aperture



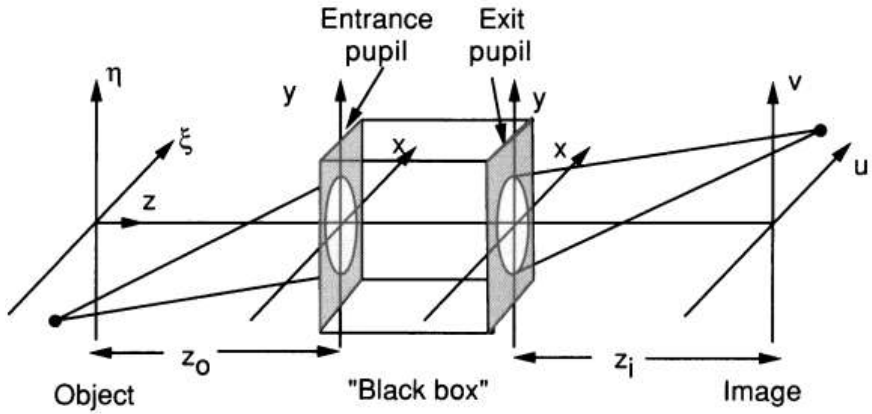
Generalized Model of Imaging Systems

- To specify the properties of the lens system, we adopt the point of view that all imaging elements may be lumped into a single "black box“
  - Significant properties of the system can be completely described by specifying only the terminal properties of the aggregate.



Generalized Model of Imaging Systems

- Two points of view that regard image resolution as being
- limited by:
  - A) the finite entrance pupil seen from the object space or
  - B) the finite exit pupil seen from the image space
- are entirely equivalent, due to the fact that these two pupils are images of each other.



The Amplitude Transfer Function (ATF)

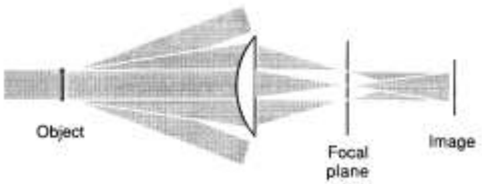
- Defined for coherent imaging systems as the Fourier transform of the space-invariant amplitude impulse response H,

Input $$G_g(f_X, f_Y) = \iint_{-\infty}^{\infty} U_g(u, v) \exp[-j2\pi(f_X u + f_Y v)]\, du\, dv$$

Output $$G_i(f_X, f_Y) = \iint_{-\infty}^{\infty} U_i(u, v) \exp[-j2\pi(f_X u + f_Y v)]\, du\, dv.$$

$$H(f_X, f_Y) = \iint_{-\infty}^{\infty} h(u, v) \exp[-j2\pi(f_X u + f_Y v)]\, du\, dv.$$

$$G_i(f_X, f_Y) = H(f_X, f_Y)\, G_g(f_X, f_Y).$$



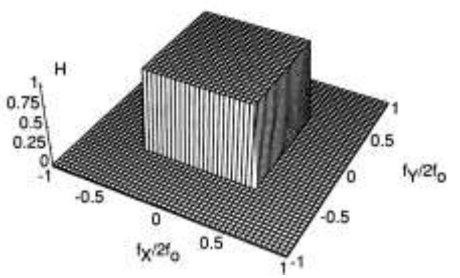
- It represents Fraunhofer diffraction and can be expressed as a scaled Fourier transform of the pupil function

$$\boxed{H(f_X, f_Y) = P(\lambda z_i f_X, \lambda z_i f_Y).}$$

Amplitude Transfer Function Examples

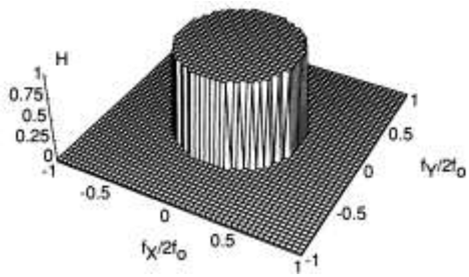
- Frequency response of diffraction-limited coherent imaging systems with square (width 2w) and circular (diameter 2w) pupils

$$P(x, y) = \text{rect}\left(\frac{x}{2w}\right)\text{rect}\left(\frac{y}{2w}\right)$$



$$P(x, y) = \text{circ}\left(\frac{\sqrt{x^2+y^2}}{w}\right).$$

$$H(f_X, f_Y) = \text{rect}\left(\frac{\lambda z_i f_X}{2w}\right)\text{rect}\left(\frac{\lambda z_i f_Y}{2w}\right)$$



$$H(f_X, f_Y) = \text{circ}\left(\frac{\sqrt{f_X^2+f_Y^2}}{w/\lambda z_i}\right).$$

The Optical Transfer Function (OTF)

- Defined for incoherent imaging systems
  - Such systems obey the intensity convolution integral:

$$I_i(u,v) = \kappa \iint_{-\infty}^{\infty} |h(u-\tilde{\xi}, v-\tilde{\eta})|^2 I_g(\tilde{\xi},\tilde{\eta})\, d\tilde{\xi}\, d\tilde{\eta}.$$

  - Normalized frequency responses of input and output given as:

$$\mathcal{G}_g(f_X, f_Y) = \frac{\iint_{-\infty}^{\infty} I_g(u,v)\exp[-j2\pi(f_X u + f_Y v)]\, du\, dv}{\iint_{-\infty}^{\infty} I_g(u,v)\, du\, dv} \qquad \mathcal{G}_i(f_X, f_Y) = \frac{\iint_{-\infty}^{\infty} I_i(u,v)\exp[-j2\pi(f_X u + f_Y v)]\, du\, dv}{\iint_{-\infty}^{\infty} I_i(u,v)\, du\, dv}.$$

$$\mathcal{H}(f_X, f_Y) = \frac{\iint_{-\infty}^{\infty} |h(u,v)|^2 \exp[-j2\pi(f_X u + f_Y v)]\, du\, dv}{\iint_{-\infty}^{\infty} |h(u,v)|^2\, du\, dv}. \qquad \mathcal{G}_i(f_X, f_Y) = \mathcal{H}(f_X, f_Y)\,\mathcal{G}_g(f_X, f_Y).$$

  - Function $\mathcal{H}$ is known as the optical transfer function (OTF) of the system. Its modulus $|\mathcal{H}|$ is known as the modulation transfer function (MTF)

Relationship Between ATF and OTF

- It can be shown that:

$$\mathcal{H}(f_X, f_Y) = \frac{\iint\limits_{-\infty}^{\infty} H\left(p + \frac{f_X}{2}, q + \frac{f_Y}{2}\right) H^*\left(p - \frac{f_X}{2}, q - \frac{f_Y}{2}\right) dp\, dq}{\iint\limits_{-\infty}^{\infty} |H(p,q)|^2\, dp\, dq}.$$

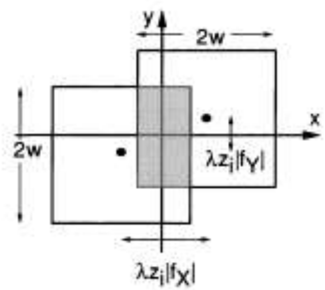
- Thus the OTF is the normalized autocorrelation function of the amplitude transfer function!

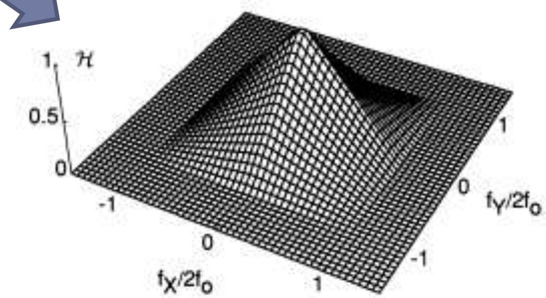
Optical Transfer Function Examples

- Consider the Frequency response of diffraction-limited incoherent imaging systems with square (width 2w) and circular (diameter 2w) pupils

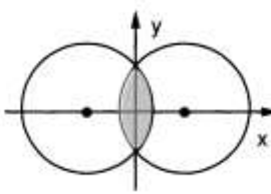
$$P(x, y) = \mathrm{rect}\left(\frac{x}{2w}\right)\mathrm{rect}\left(\frac{y}{2w}\right)$$

$$P(x, y) = \mathrm{circ}\left(\frac{\sqrt{x^2 + y^2}}{w}\right).$$

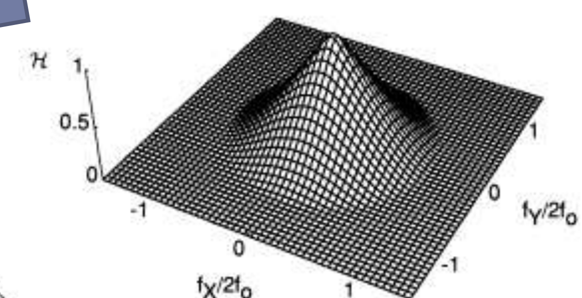


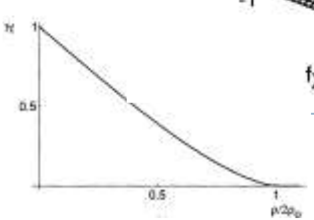


$$\mathcal{H}(f_X, f_Y) = \Lambda\left(\frac{f_X}{2f_o}\right)\Lambda\left(\frac{f_Y}{2f_o}\right)$$



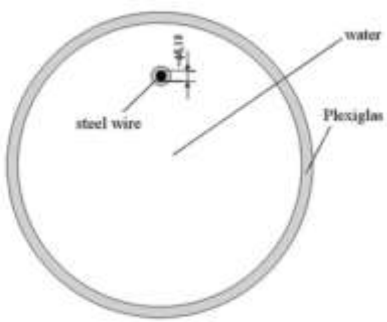
$$\mathcal{H}(\rho) = \begin{cases} \frac{2}{\pi}\left[\arccos\left(\frac{\rho}{2\rho_o}\right) - \frac{\rho}{2\rho_o}\sqrt{1 - \left(\frac{\rho}{2\rho_o}\right)^2}\right] & \rho \le 2\rho_o \\ 0 & \text{otherwise.} \end{cases}$$

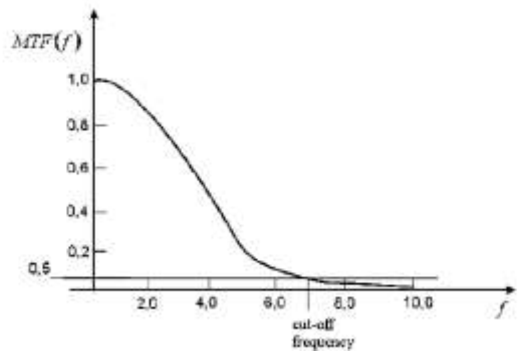




MTF in Computed Tomography

- Spatial Resolution of CT is defined using MTF
  - MTF defines the frequency domain relationship between the original and the reconstructed image in the presence of noise, and determines the ability of the scanner to capture rapidly changing attenuation coefficients in the object
  - Spatial resolution is often defined in terms of the cut-off frequency of one-dimensional transfer function, i.e. the value at which the function MTF(f) drops to the 50, 10 or 2% level
  - Measured using special phantom: impulse response





Exercise

- Solve problems: 2.1, 2.6, 2.10, 2.11, 2.13, 3.5, 4.7, 4.9, 4.10
- Report a typical MTF of a medical imaging modality of your choice