Medical Image Reconstruction Term II – 2012

Topic 6: Tomography

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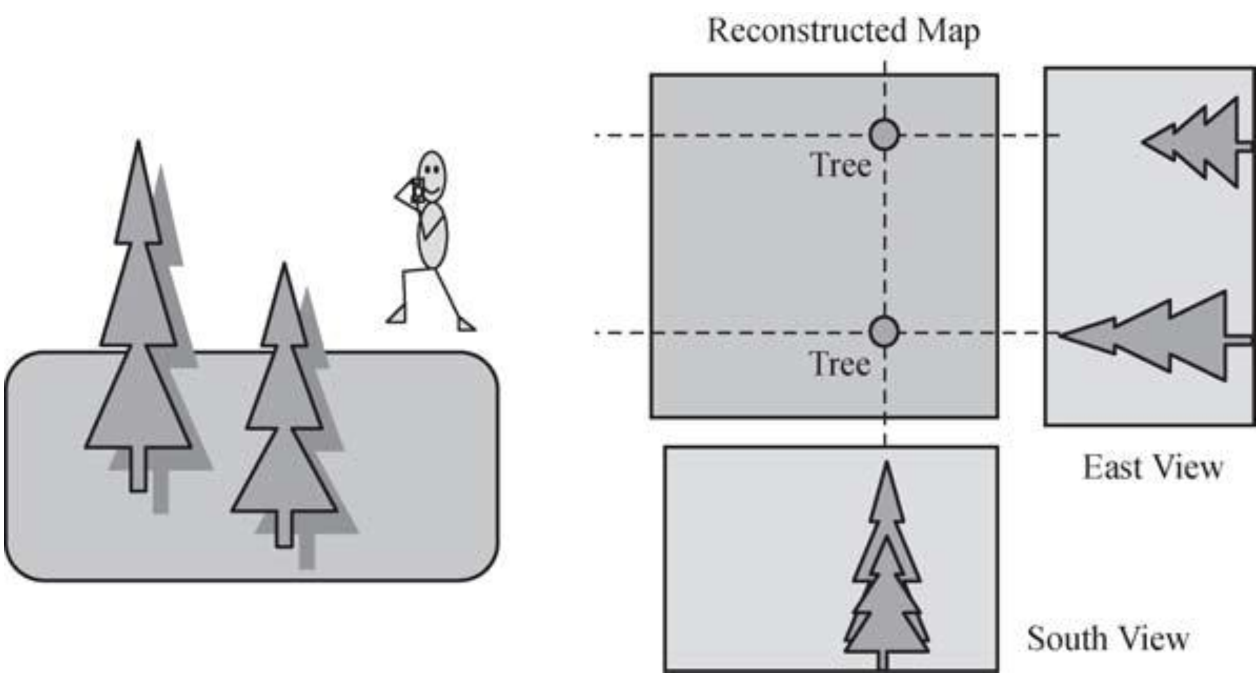
Tomography

- The Greek word "*tomos*" means a section, a slice, or a cut.
- Tomography is the process of imaging a cross section
- Particularly useful in medical imaging
  - Nobody wants to be cut open to see what is inside!

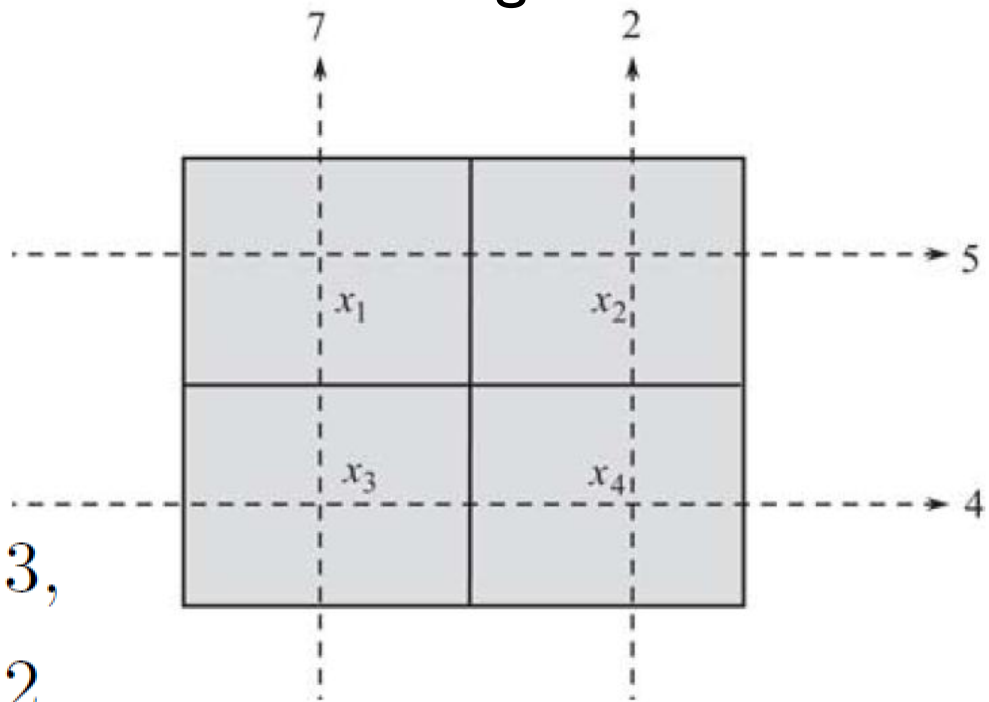
Example

- Can you compute the locations of the trees from 2 images?
  - Answer: Yes



Example

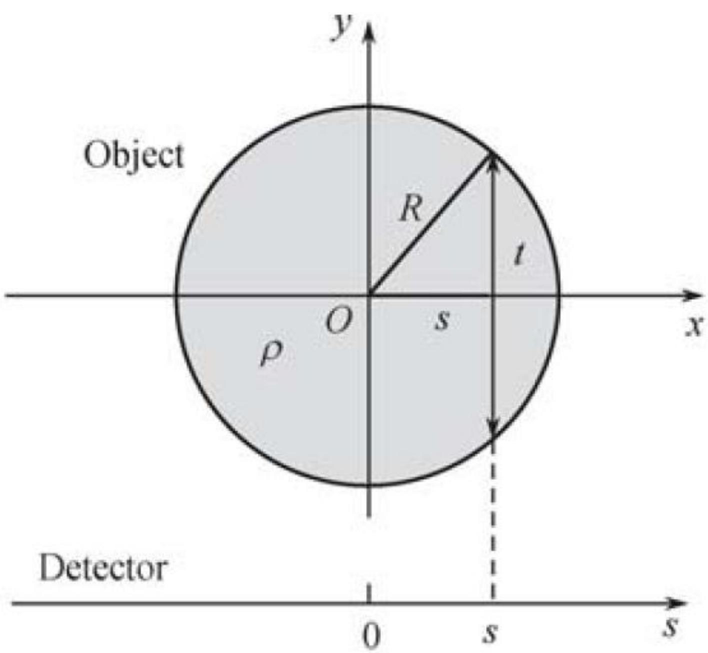
- Can you compute the values of this matrix given its projections?
  - Answer: Yes

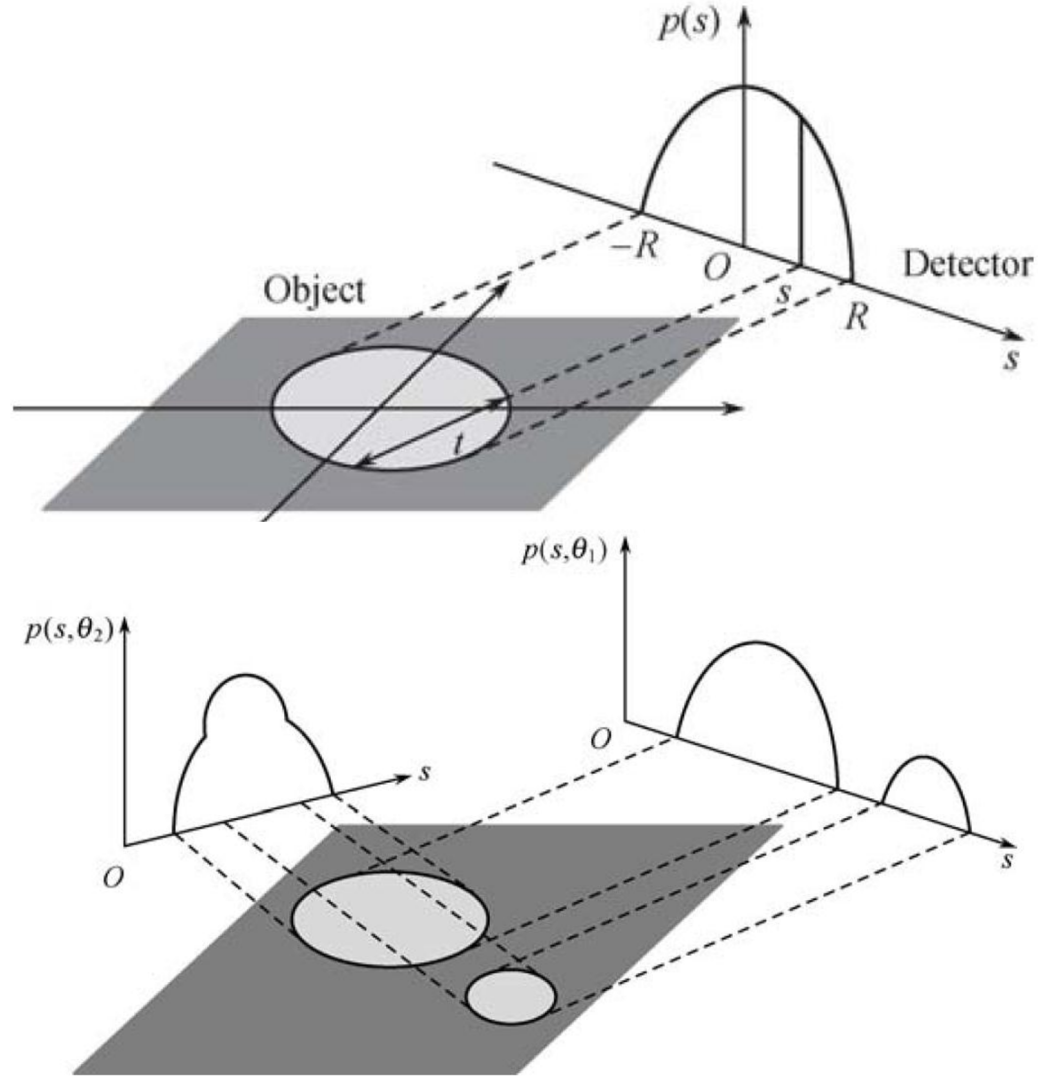


$$\left.\begin{aligned} x_1 + x_2 &= 5, \\ x_3 + x_4 &= 4, \\ x_1 + x_3 &= 7, \\ x_2 + x_4 &= 2. \end{aligned}\right\} \qquad \begin{aligned} x_1 &= 3, \\ x_2 &= 2, \\ x_3 &= 4, \\ x_4 &= 0. \end{aligned}$$

Projection

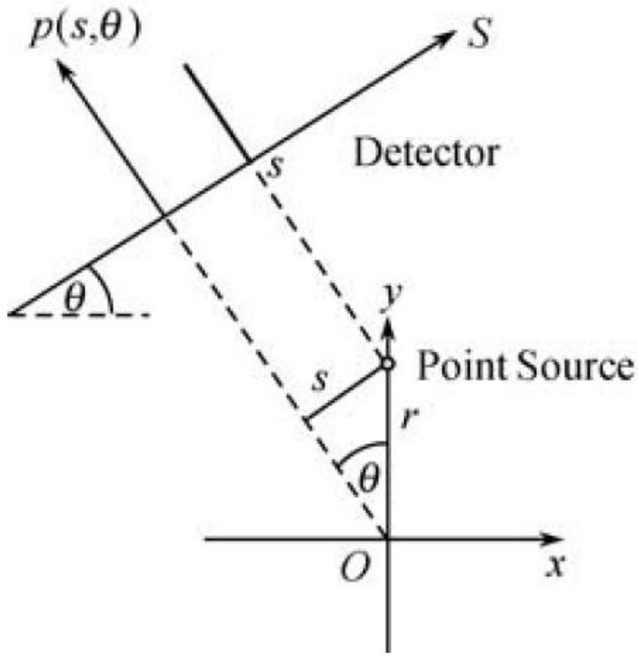
- Also termed *ray sum*, *line integral*, or *Radon transform*



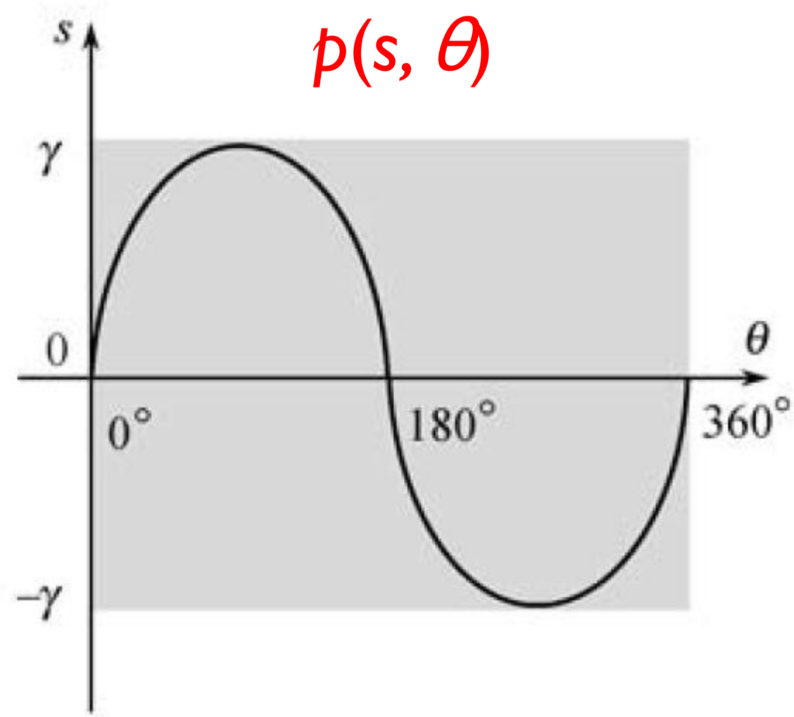


Sinogram

- Displays angle dependence of projections
  - Example: point source on the *y*-axis to further illustrate the angle θ dependency of the projection $p(s, \theta)$

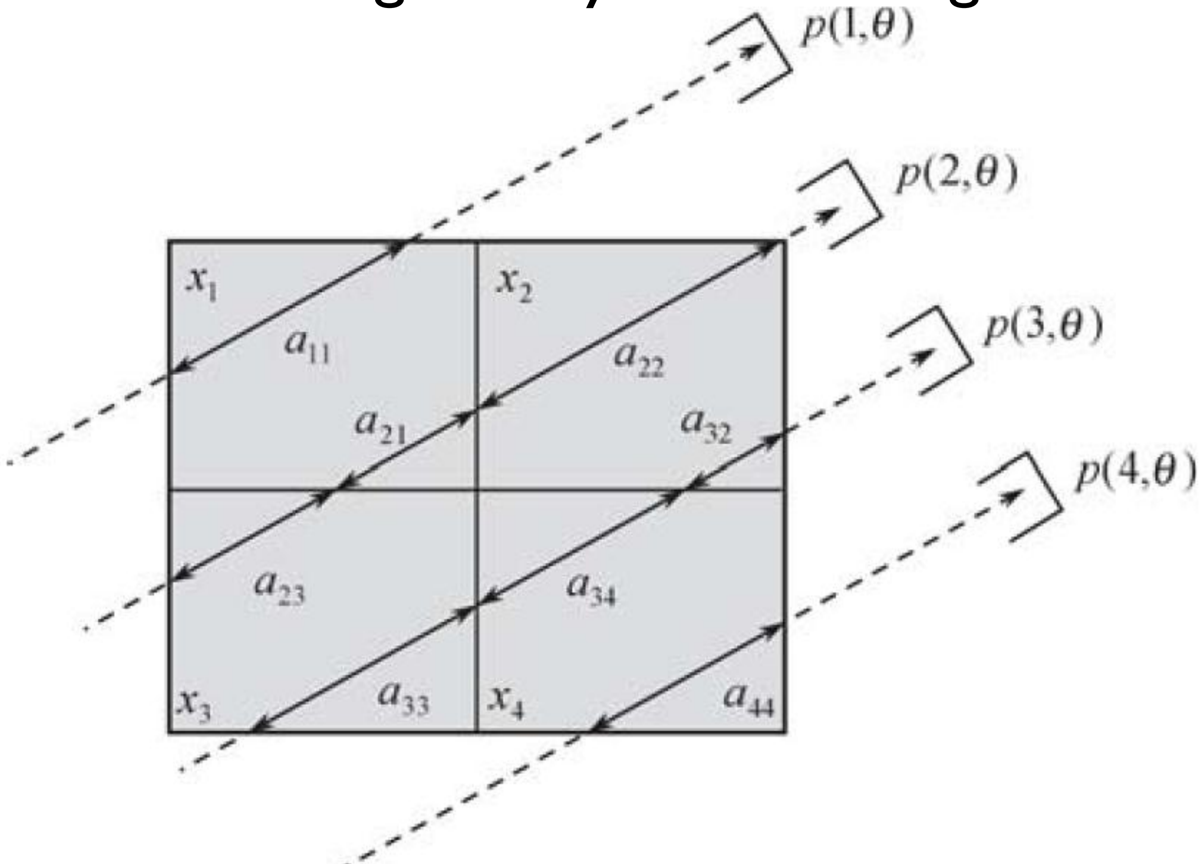


Sinogram
$p(s, \theta)$



Projection of Discrete Object

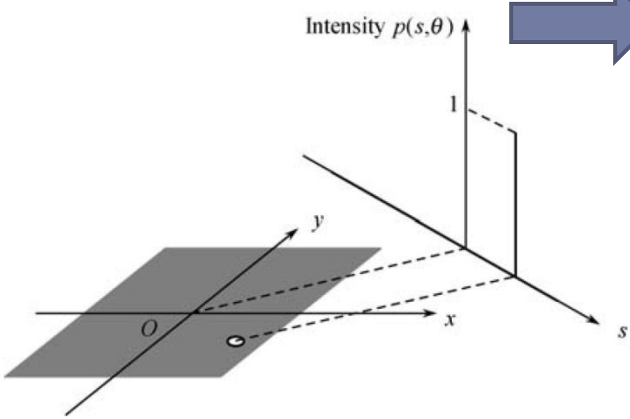
- Projections are weighted by the line-length within each pixel

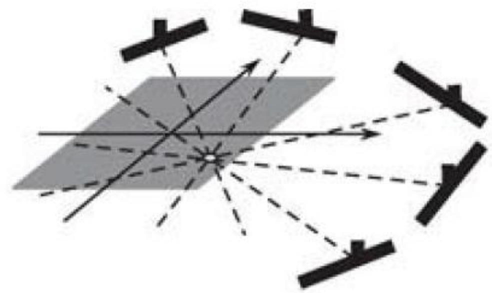


$$p(i,\theta) = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4, \qquad i = 1, 2, 3, 4.$$

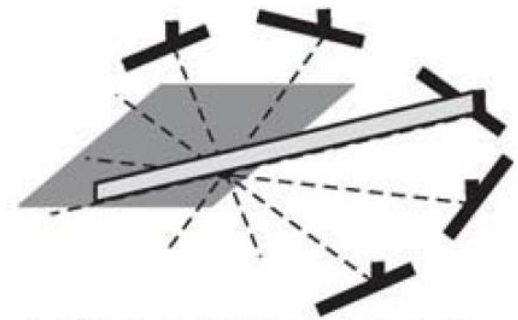
Image Reconstruction of a Point Source

- In image reconstruction, we not only need to find the location but also the intensity value of the object of interest
  - Backprojection

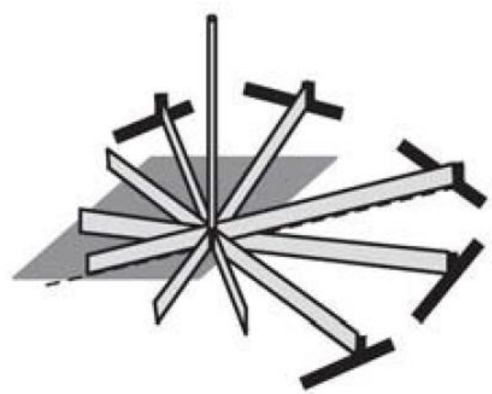




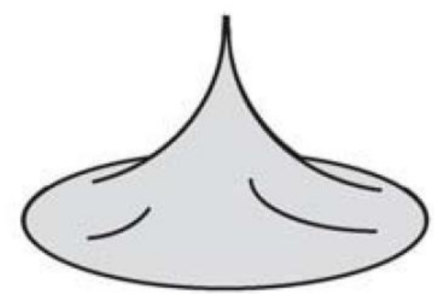
(a) Project a point source



(b) Backproject from one view



(c) Backproject from a few views

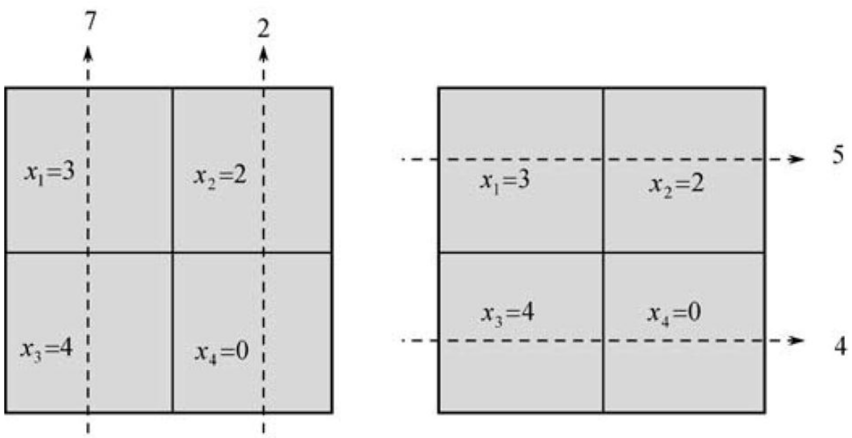


(d) Backproject from all views

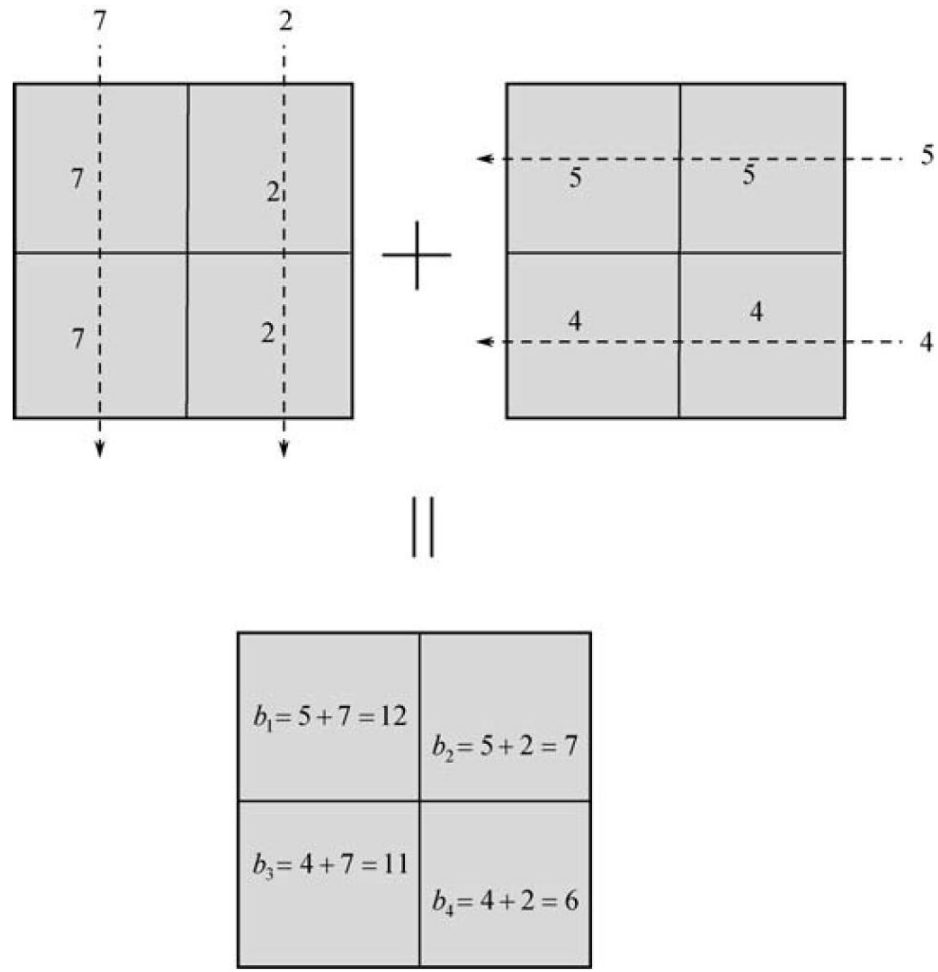
Backprojection Example

- Blurred version of original

Projection



Backprojection

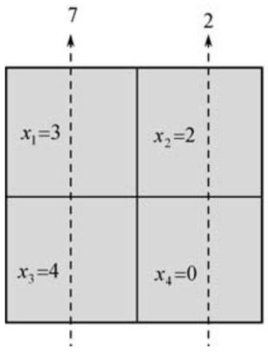


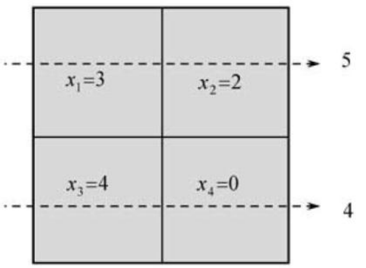
Backprojection in Algebraic Form

- Projection P

$$P = AX, \quad X = [x_1, x_2, x_3, x_4]^{\mathrm{T}}$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$





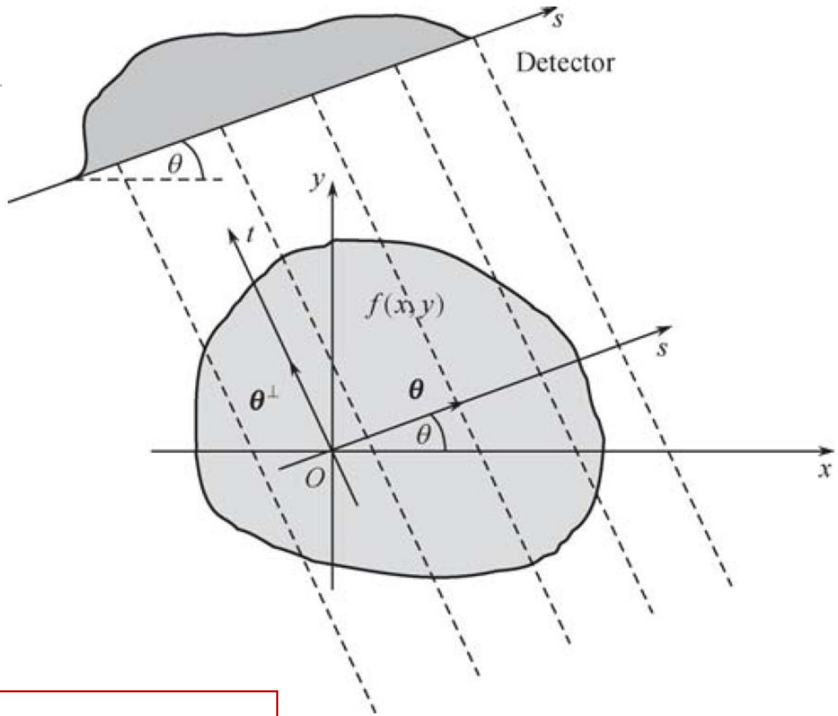
$$P = [p(1, 0°), p(2, 0°), p(1, 270°), p(2, 270°)]^{\mathrm{T}} = [7, 2, 5, 4]^{\mathrm{T}}.$$

- Backprojection: use adjoint operator A^T

$$B = A^{\mathrm{T}}P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 7 \\ 2 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \\ 11 \\ 6 \end{bmatrix}$$

Radon Transform

- Computes parallel projections at specific angles



$$f(x,y) \rightarrow p(s,\theta)$$

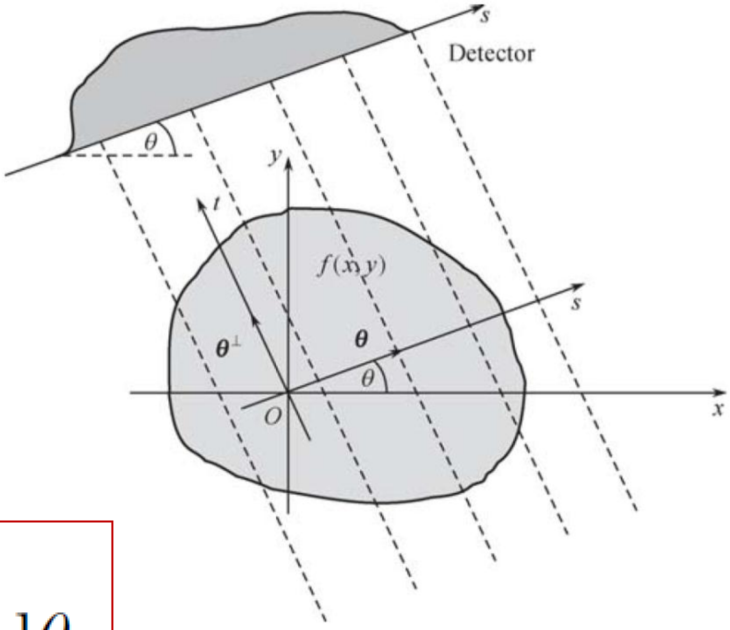
$$p(s,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - s)\mathrm{d}x\mathrm{d}y,$$

$$p(s,\theta) = \int_{-\infty}^{\infty} f(s\cos\theta - t\sin\theta, s\sin\theta + t\cos\theta)\mathrm{d}t,$$

$$\mathscr{R}f(t,\theta) := \int_{\ell_{t,\theta}} f\,ds = \int_{s=-\infty}^{\infty} f(t\cos(\theta) - s\sin(\theta), t\sin(\theta) + s\cos(\theta))\,ds.$$

Inverse Radon Transform

- Parallel Beam Backprojection



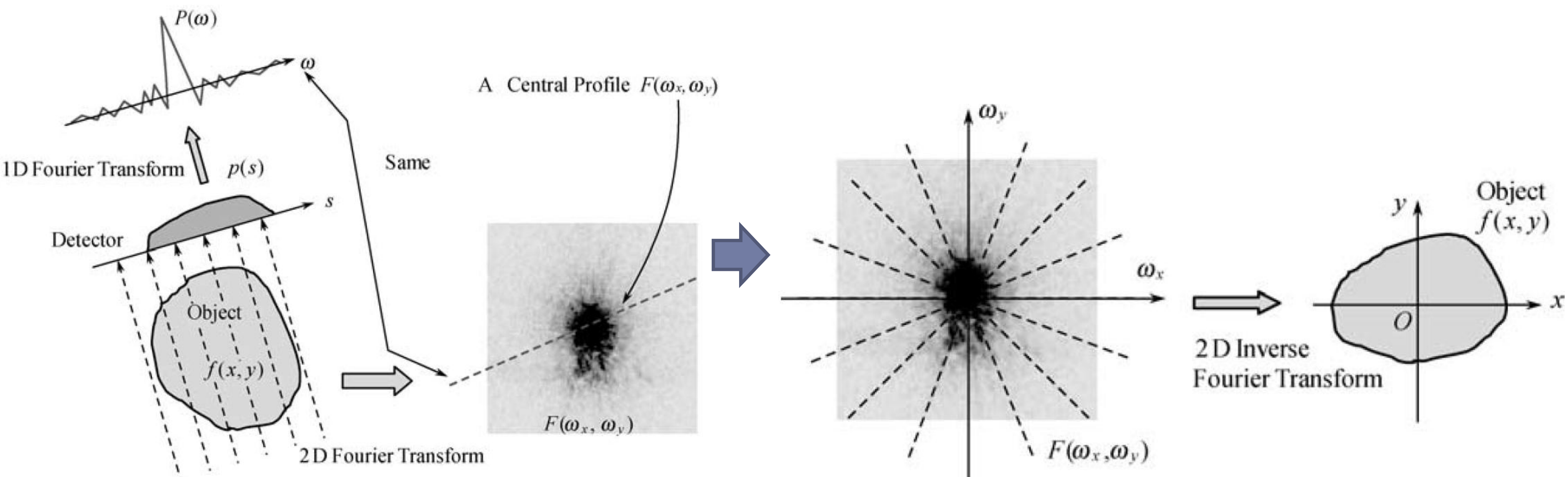
$$p(s,\theta) \rightarrow f(x,y)$$

$$b(x,y) = \int_0^{\pi} p(s,\theta)\big|_{s=x\cos\theta+y\sin\theta} \mathrm{d}\theta,$$

$$b(x,y) = \int_0^{\pi} p(\boldsymbol{x}\cdot\boldsymbol{\theta},\theta)\mathrm{d}\theta,$$

Backprojection in Frequency Domain

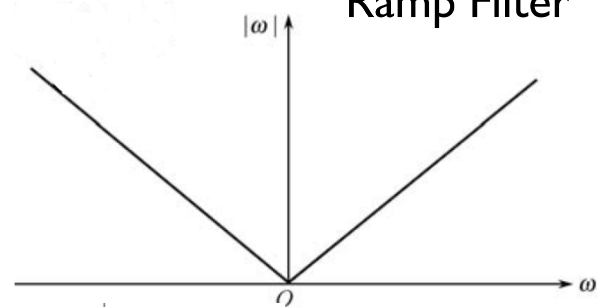
- Each view adds a line in the Fourier space
  - Central area in k-space has higher sample density and results in effective lowpass filtered spectrum



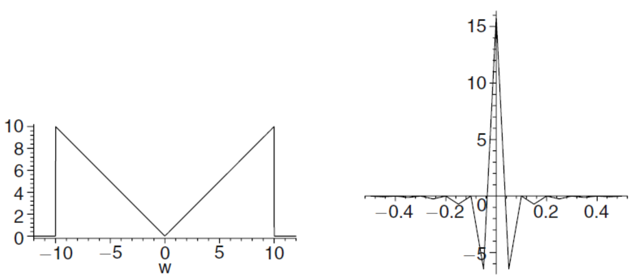
Filtered Backprojection

- To counter this blurring effect, we must compensate for the non-uniformity in the Fourier space
  - Density in the Fourier space is proportional to: $\frac{1}{\sqrt{\omega_x^2 + \omega_y^2}}$.
  - Solution: multiply k-space by ramp filter, $\sqrt{\omega_x^2 + \omega_y^2}$
    - Method 1: Filter individual projections
    - Method 2: Filter whole Image
- Practical Realization

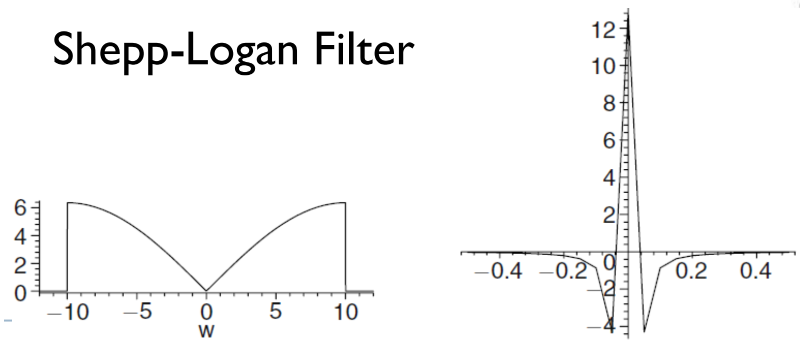
Ramp Filter



Ram-Lak Filter



Shepp-Logan Filter



Hilbert Transform Based Formulation

$$\mathscr{F}\left(\frac{df}{dx}\right)(\omega) = i\omega\mathscr{F}(f)(\omega). \Rightarrow \mathscr{F}\left(\frac{\partial(\mathscr{R}f)(t,\theta)}{\partial t}\right)(S,\theta) = iS\mathscr{F}(\mathscr{R}f)(S,\theta).$$

$$|S| = S \cdot \operatorname{sgn}(S)$$

$$i \cdot \operatorname{sgn}(S) \cdot \mathscr{F}\left(\frac{\partial(\mathscr{R}f)(t,\theta)}{\partial t}\right)(S,\theta) = -|S|\mathscr{F}(\mathscr{R}f)(S,\theta).$$

$$f(x,y) = \frac{-1}{2}\mathscr{B}\left\{\mathscr{F}^{-1}\left[i \cdot \operatorname{sgn}(S) \cdot \mathscr{F}\left(\frac{\partial(\mathscr{R}f)(t,\theta)}{\partial t}\right)(S,\theta)\right]\right\}(x,y).$$

Define Hilbert Transform as: $\mathscr{H}g(t) = \mathscr{F}^{-1}\left[i \cdot \operatorname{sgn}(\omega) \cdot \mathscr{F}g(\omega)\right](t).$

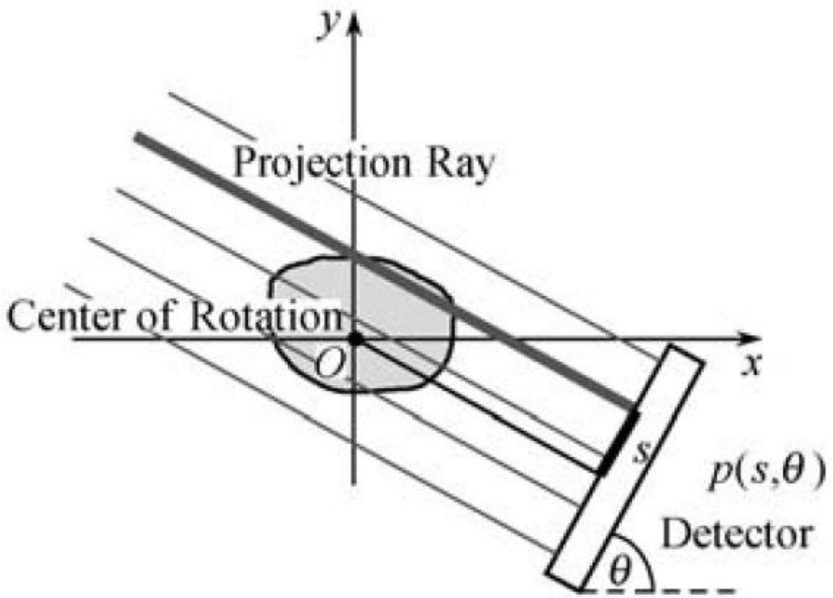
$$\Rightarrow \boxed{f(x,y) = \frac{-1}{2}\mathscr{B}\left[\mathscr{H}\left(\frac{\partial(\mathscr{R}f)(t,\theta)}{\partial t}\right)(S,\theta)\right](x,y).}$$

Parallel Beam Reconstruction Methods

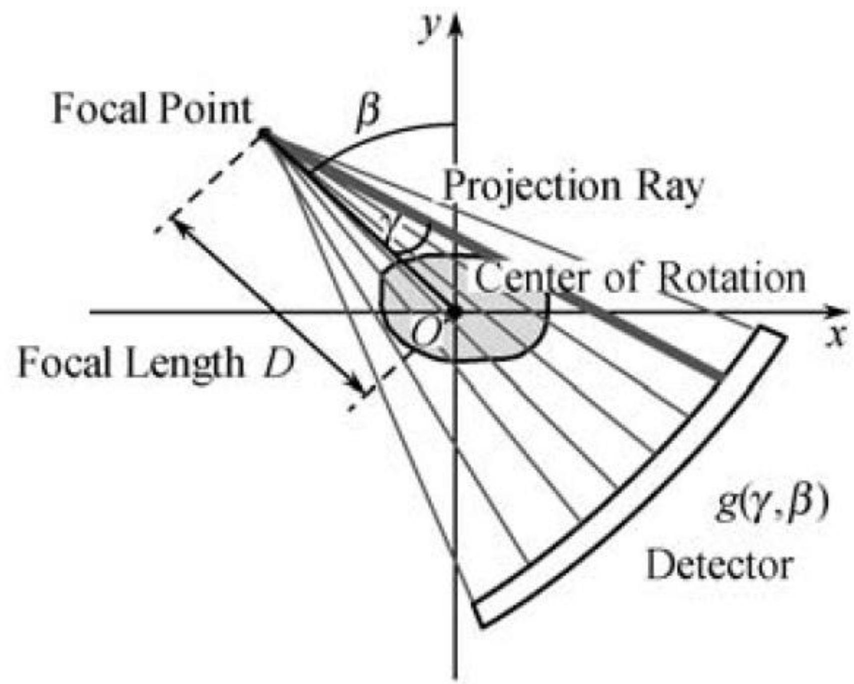
Method	Step 1	Step 2	Step 3
1	1D Ramp filter with Fourier transform	Backprojection	---
2	1D Ramp filter with convolution	Backprojection	
4	Backprojection	2D Ramp filter with Fourier transform	
	Backprojection	2D Ramp filter with 2D convolution	
3	Derivative	Hilbert transform	Backprojection
5	Derivative	Backprojection	Hilbert transform
	Backprojection	Derivative	Hilbert transform
	Hilbert transform	Derivative	Backprojection
	Hilbert transform	Backprojection	Derivative
	Backprojection	Hilbert transform	Derivative

Fan Beam Reconstruction Problem

- Almost all present CT systems use fan beam rather than parallel beam projections
  - Much more efficient: faster acquisition, lower patient dose



(a) Parallel Beam

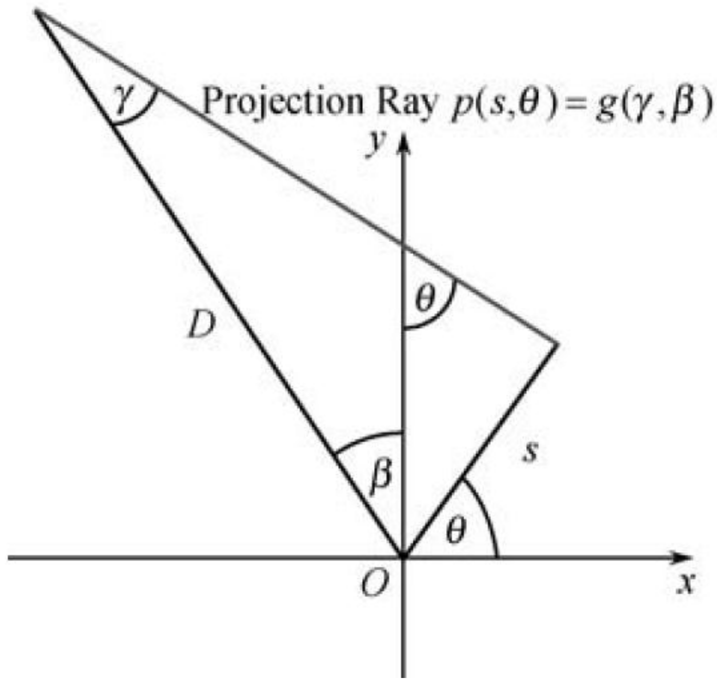


(b) Fan Beam

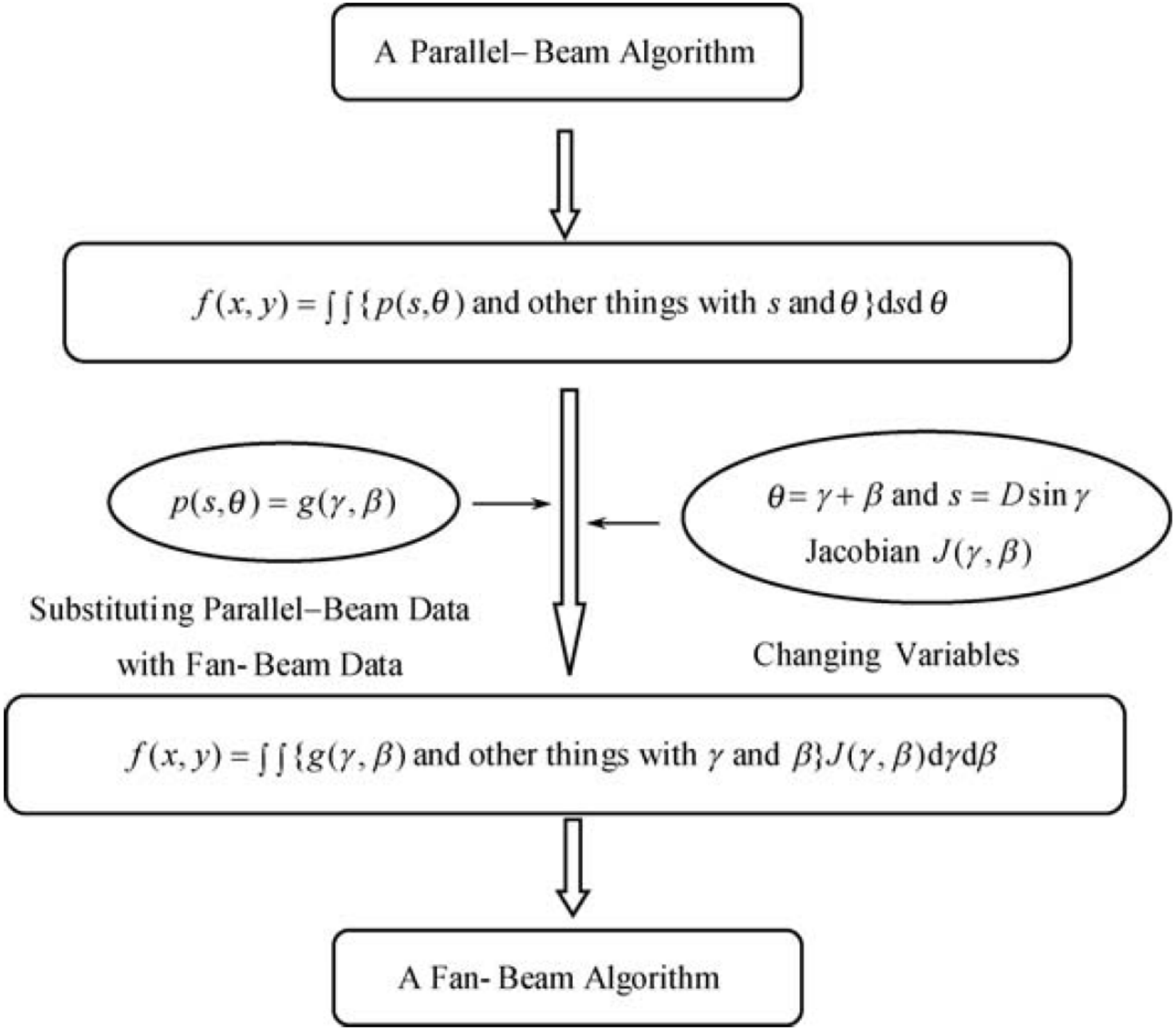
Fan Beam Reconstruction Problem

- rebin every fan-beam ray into a parallel-beam ray. For each fan-beam ray-sum $g(\gamma, \beta)$, we can find a parallel beam ray-sum $p(s, \theta)$ that has same orientation as the fan-beam ray with the relations,

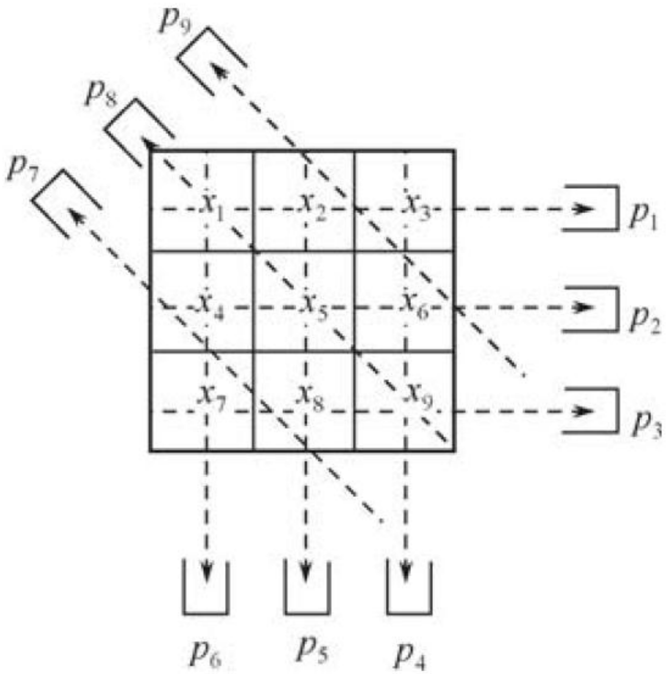
$$\theta = \gamma + \beta, \quad s = D \sin \gamma, \quad \Rightarrow \quad p(s, \theta) = g(\gamma, \beta).$$



Fan Beam Reconstruction Problem



Algebraic Reconstruction Technique



$$
\begin{cases}
x_1 + x_2 + x_3 = p_1, \\
x_4 + x_5 + x_6 = p_2, \\
x_7 + x_8 + x_9 = p_3, \\
x_3 + x_6 + x_9 = p_4, \\
x_2 + x_5 + x_8 = p_5, \\
x_1 + x_4 + x_7 = p_6, \\
2(\sqrt{2}-1)x_4 + (2-\sqrt{2})x_7 + 2(\sqrt{2}-1)x_8 = p_7, \\
\sqrt{2}x_1 + \sqrt{2}x_5 + \sqrt{2}x_9 = p_8, \\
2(\sqrt{2}-1)x_2 + (2-\sqrt{2})x_3 + 2(\sqrt{2}-1)x_6 = p_9.
\end{cases}
$$

$$AX = P, \qquad X = A^+P = V\Sigma^+U^\mathrm{T}P.$$

$$
\begin{aligned}
&\boldsymbol{x}^{next} \\
&= \boldsymbol{x}^{current} - Backproject_{ray}\left\{\frac{Project_{ray}(\boldsymbol{x}^{current}) - Measurement_{ray}}{Normalization\ Factor}\right\}.
\end{aligned}
$$

Other Algebraic Methods

- Gradient descent

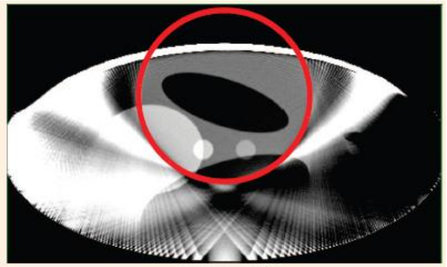
$$\boldsymbol{x}^{next} = \boldsymbol{x}^{current} - a_{current} \boldsymbol{\Delta}(\boldsymbol{x}^{current}),$$

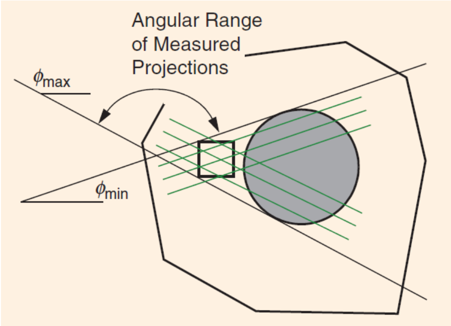
- Maximum-Likelihood Expectation-Maximization

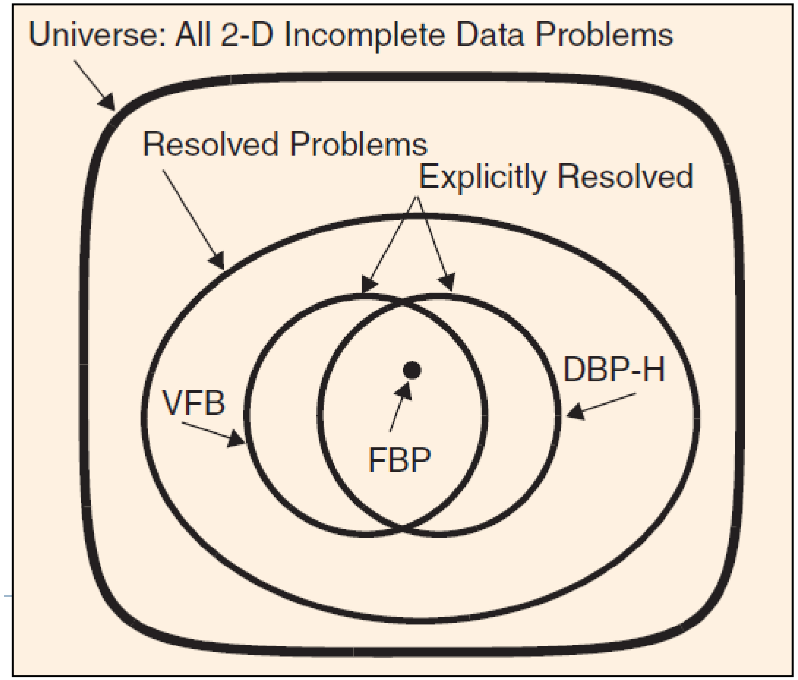
$$\boldsymbol{x}^{next} = \boldsymbol{x}^{current} \frac{Backproject \left\{ \dfrac{Measurement}{Project \left(\boldsymbol{x}^{current}\right)} \right\}}{Backproject \left\{ \mathbf{1} \right\}},$$

Advanced Tomography Problems

- Reconstruction from incomplete data
  - Truncated projections
  - Limited-angle projections
  - Exterior data
  - ROI reconstruction
- Extension to 3D
- Stability of reconstruction







Matlab Functions

- Look up the help for the following function:
  - radon
  - iradon
  - fan2para
  - fanbeam
  - ifanbeam
  - para2fan
  - phantom

Exercise

- P1. For a medical image of your choice:
  - A. Generate the Radon transform and display its sinogram
  - B. Reconstruct the image back from its projections
  - C. Compare the two images and record the error for different numbers of projections and provide your comments.
- P2. Repeat the above problem P1 for a fan beam system (rather than the parallel beam system in P1). Also, compare the parallel beams obtained from rebinning the fan rays and comment on what you found.
- P3. Compare different filtering strategies and provide your choice of the best methodology based on an experimental study. (study filter type and support, 1D vs. 2D implementation, etc.)
- P4. Compare the parallel beam projections obtained from a Shepp-Logan phantom to the ones you generate using the analytical Shepp-Logan phantom and comment on the results.
- P5. Do a literature review on ONE advanced tomography problem and come up with a 1 page summary of the state of the art and a comprehensive list of references.