Medical Image Reconstruction
Term II – 2010

Topic 1: Mathematical Basis
Lecture 3

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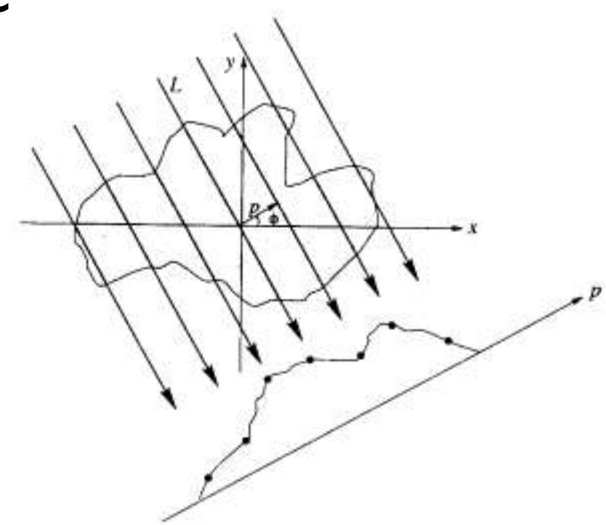
Topics Today

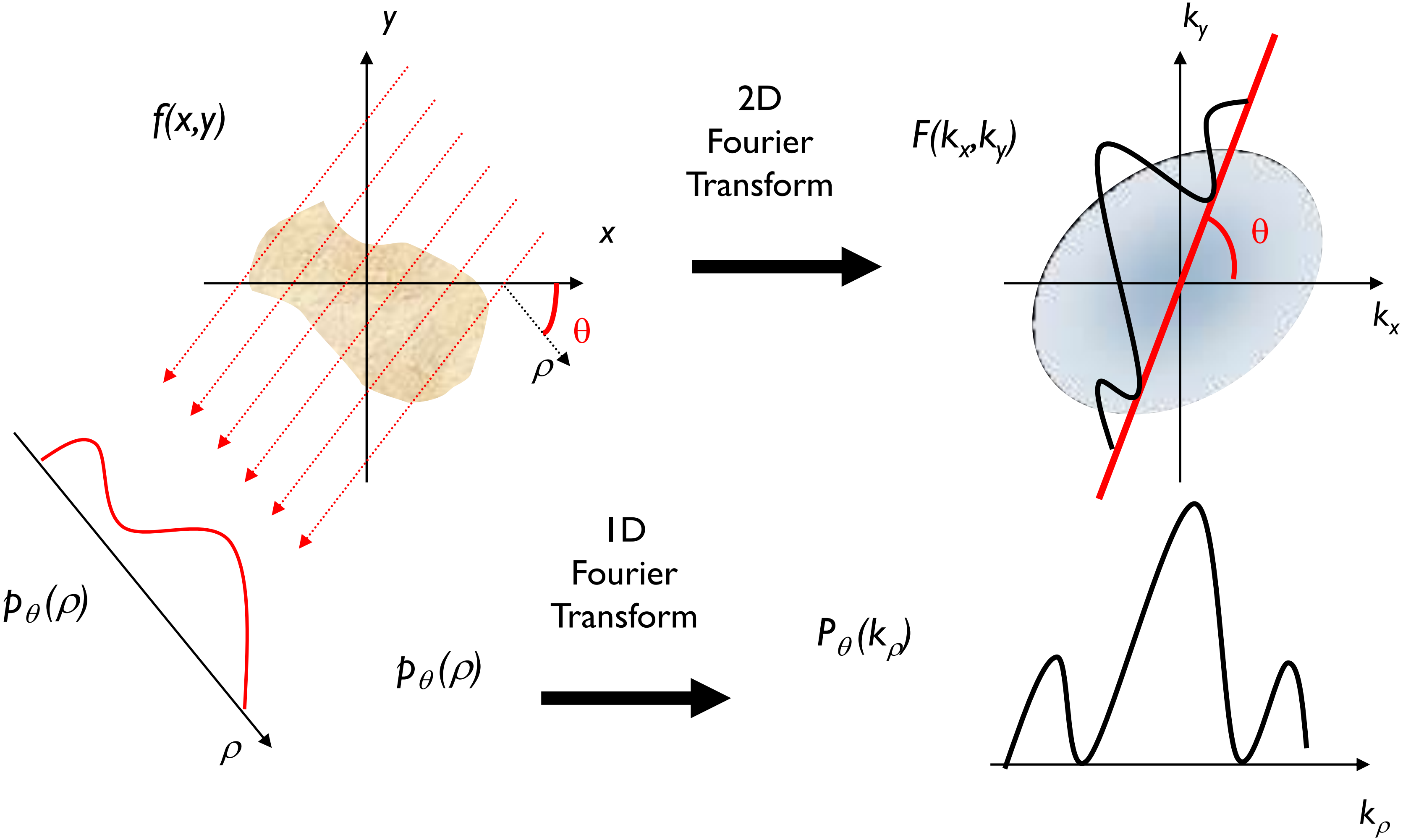
- Projection-slice theorem
- Interlaced Fourier transform

Projection-Slice Theorem

- Also known as Central-Slice Theorem
- A property of the Fourier transform
- Relates the projection data in the spatial domain to the frequency domain
- States that the 1D Fourier transform of the projection of an image at an angle θ is equal to the slice of the 2D Fourier transform at the same angle



Projection-Slice Theorem



Projection-Slice Theorem

- 2D Fourier transformation:

$$F(k_x, k_y) = \iint f(x,y) \cdot e^{-j2\pi(k_x \cdot x + k_y \cdot y)} dxdy$$

- The slice of the 2D Fourier transform at k_x=0 is given by:

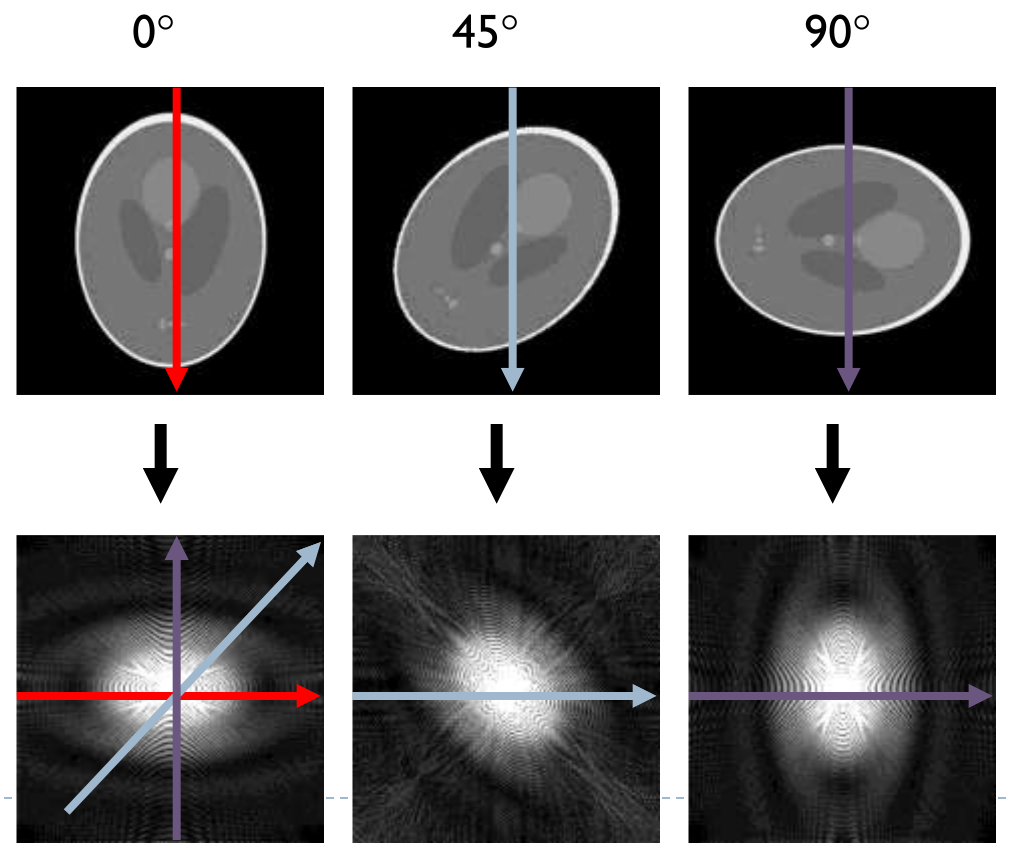
$$F(0, k_y) = \int \left(\int f(x,y) dx \right) e^{-j2\pi k_y \cdot y} dy$$

and at k_y=0 is given by

$$F(k_x, 0) = \int \left(\int f(x,y) dy \right) e^{-j2\pi k_x \cdot x} dx$$

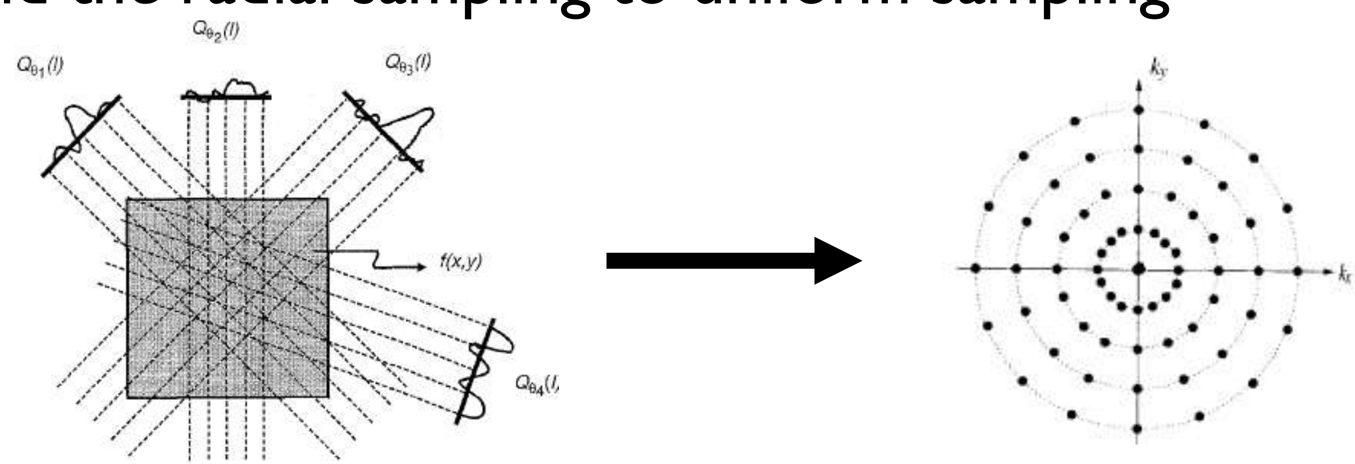
Projection-Slice Theorem

- For a general angle, the rotation property of the Fourier transformation can be used to generalize the mathematical result for a vertical projection to any angle



Projection-Slice Theorem: Application to CT

- The projection data can be shown to correspond to radial sampling of the frequency domain
- It is not straightforward to numerically compute the image from this frequency domain representation
  - Limitation of the DFT to uniform sampled data
- Interpolation can be used in the frequency domain to re-grid the radial sampling to uniform sampling

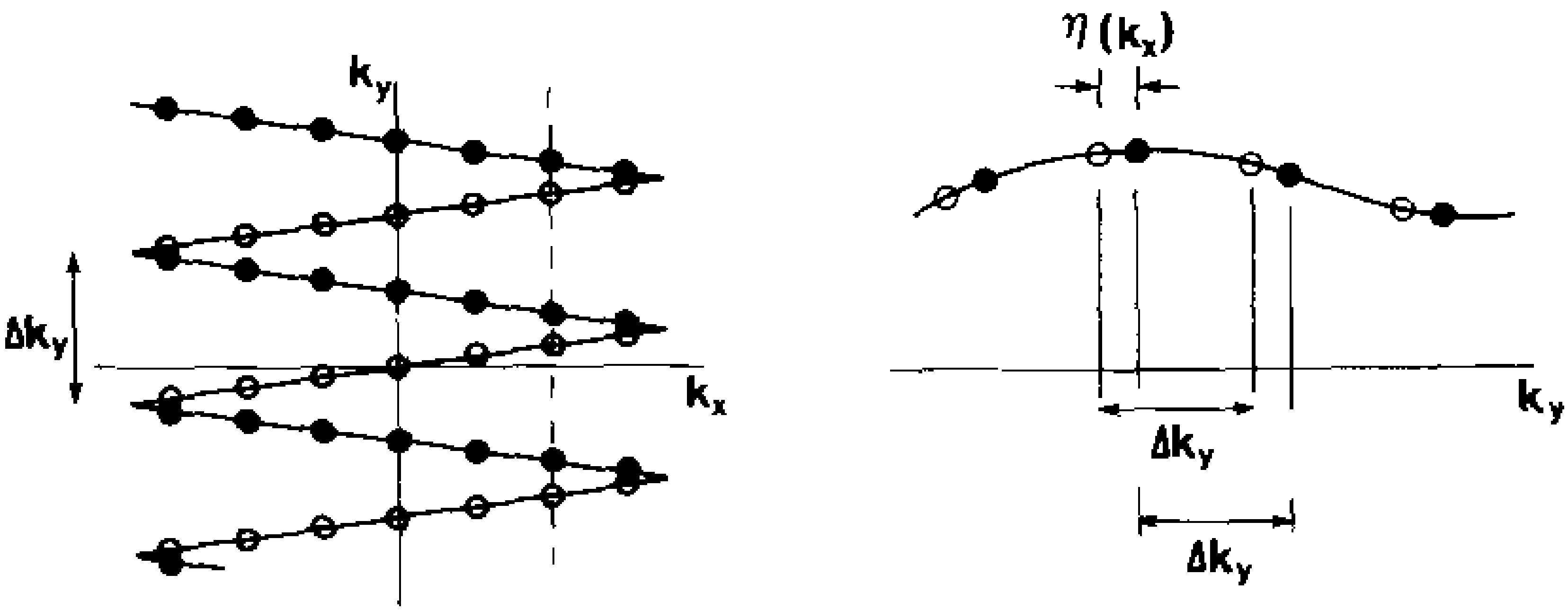


Projection-Slice Theorem: Application to MRI

- Navigator echo motion estimation
  - Acquire a single k-space line in the middle to estimation linear translation in this direction
- Early MRI reconstruction based on backprojection algorithms

Interlaced Fourier Transform

- A special case of nonuniform Fourier transform



Interlaced Fourier Transform

- Mathematical formulation

$$g(k_x, y) = \frac{-e^{\kappa 2\pi i \xi(k_x)}}{1 - e^{\kappa 2\pi i \xi(k_x)}} g_P(k_x, y) + \frac{e^{iy\eta(k_x)}}{1 - e^{\kappa 2\pi i \xi(k_x)}} g_N(k_x, y),$$

$$h(k_x, y) = \frac{1}{1 - e^{\kappa 2\pi i \xi(k_x)}} g_P(k_x, y) - \frac{e^{iy\eta(k_x)}}{1 - e^{\kappa 2\pi i \xi(k_x)}} g_N(k_x, y),$$

$$G(k_x, y) = h\left(k_x, y + \frac{L_y}{2}\right) \qquad -\frac{L_y}{2} \leqslant y < -\frac{L_y}{4},$$

$$G(k_x, y) = g(k_x, y) \qquad -\frac{L_y}{4} \leqslant y \leqslant \frac{L_y}{4},$$

$$G(k_x, y) = h\left(k_x, y - \frac{L_y}{2}\right) \qquad \frac{L_y}{4} < y \leqslant \frac{L_y}{2}.$$

Interlaced Fourier Transform

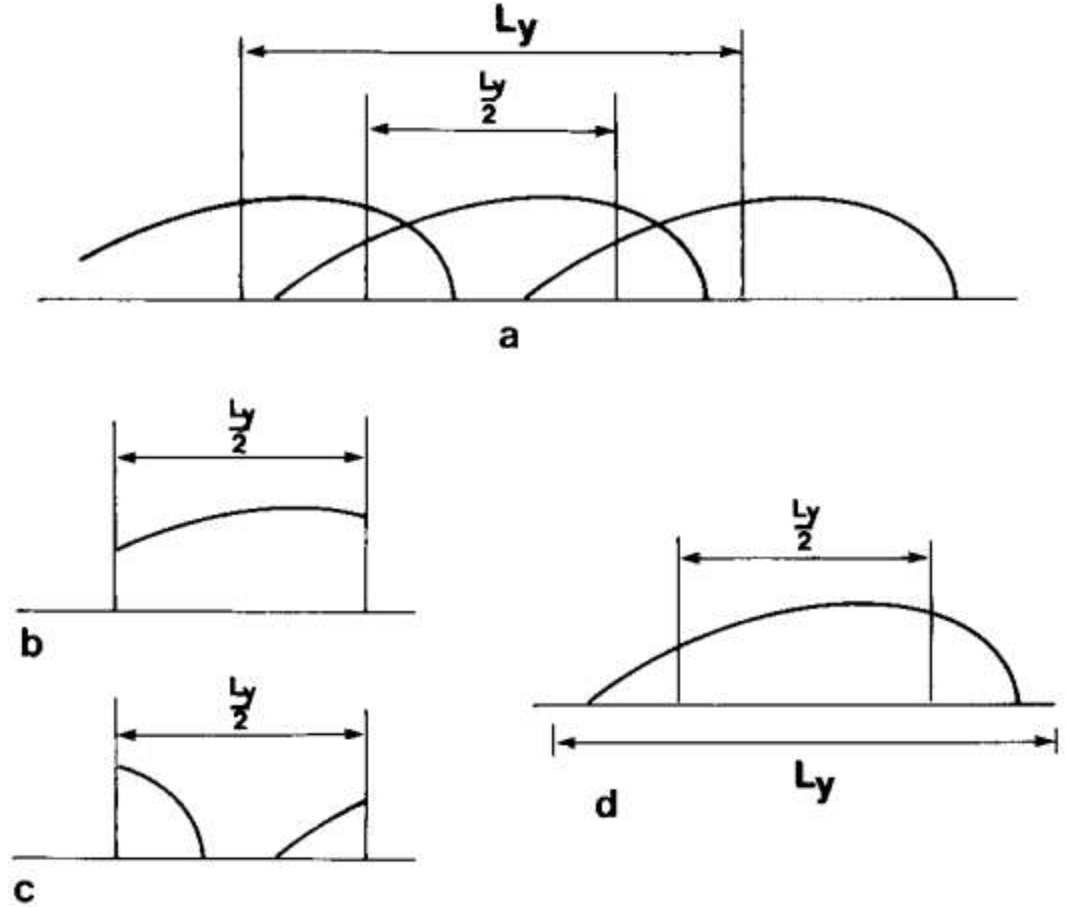


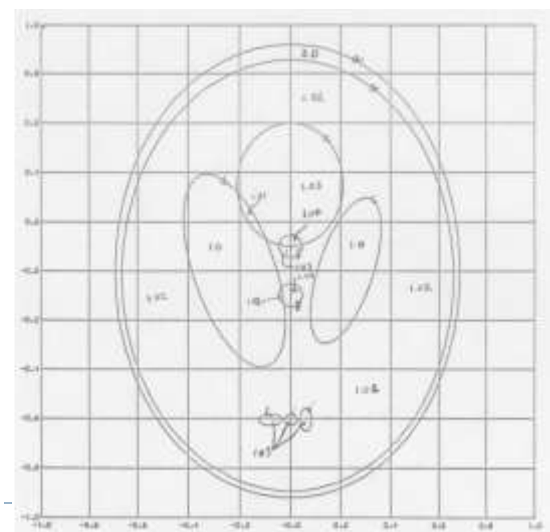
FIG. 2. (a) $g_P(k_x, y)$ at a specific k_x. (b) $g(k_x, y)$ at same k_x as in (a). (c) $h(k_x, y)$ at same k_x as in (a). (d) $G(k_x, y)$ at same k_x as in (a), (b), and (c).

Shepp-Logan Phantom

- Numerical phantom used to simulate the human head to evaluate reconstruction algorithms in computed tomography

RECONSTRUCTING INTERIOR HEAD TISSUE
FROM X-RAY TRANSMISSIONS

L. A. Shepp and B. F. Logan
Bell Laboratories
Murray Hill, New Jersey 07974



Shepp-Logan Phantom

TABLE 1

Ellipses	Center	Major Axis	Minor Axis	Theta	Gray level
a	(0,0)	.69	.92	0	2
b	(0,-.0184)	.6624	.874	0	-.98
c	(.22,0)	.11	.31	-18°	-.02
d	(-.22,0)	.16	.41	18°	-.02
e	(0,.35)	.21	.25	0	.01
f	(0,.1)	.046	.046	0	.01
g	(0,-.1)	.046	.046	0	.02
h	(-.08,-.605)	.046	.023	0	.01
i	(0,-.605)	.023	.023	0	.01
j	(.06,-.605)	.023	.046	0	.01

Shepp-Logan Phantom: 3D

Table 1: 3D Shepp-Logan Phantom Specification for MRI

Ellipsoid (i)	Center (r_0)			Half-Axis			Angle	Spin Density	Portion Subtracted	Tissue Type
	x	y	z	a	b	c	□			
1*	0	0	0	0.72	0.95	0.93	0	0.8	None	Scalp
2	0	0	0	0.69	0.92	0.9	0	0.12 [13]	**2**[Prop[**1**]]	Bone & Marrow
3*	0	-0.0184	0	0.6624	0.874	0.88	0	0.98 [13]	**3**[Prop[**2**]]	CSF
4**	0	-0.0184	0	0.6524	0.864	0.87	0	0.745 [14]	**4**[Prop[**3**]]	Gray Matter
5	-0.22	0	-0.25	0.41	0.16	0.21	-72^0	0.98	**5**[Prop[**4**]]	CSF
6	0.22	0	-0.25	0.31	0.11	0.22	72^0	0.98	**6**[Prop[**4**]]	CSF
7	0	0.35	-0.25	0.21	0.25	0.35	0	0.617 [14]	**7**[Prop[**4**]]	White Matter
8	0	0.1	-0.25	0.046	0.046	0.046	0	0.95 [6]	**8**[Prop[**4**]]	Tumor
9	-0.08	-0.605	-0.25	0.046	0.023	0.02	0	0.95	**9**[Prop[**4**]]	Tumor
10	0.06	-0.605	-0.25	0.046	0.023	0.02	-90^0	0.95	**10**[Prop[**4**]]	Tumor
11	0	-0.1	-0.25	0.046	0.046	0.046	0	0.95	**11**[Prop[**4**]]	Tumor
12	0	-0.605	-0.25	0.023	0.023	0.023	0	0.95	**12**[Prop[**4**]]	Tumor
13†*	0.06	-0.105	0.0625	0.056	0.04	0.1	-90^0	0.93 [6]	**13**[Prop[**4**]]	Tumor
14†*	0	0.1	0.625	0.056	0.056	0.1	0	0.98	**14**[Prop[**4**]]	CSF
15††	0.56	-0.4	-0.25	0.2	0.03	0.1	70^0	0.85 [15]	Not Used	Blood Clot

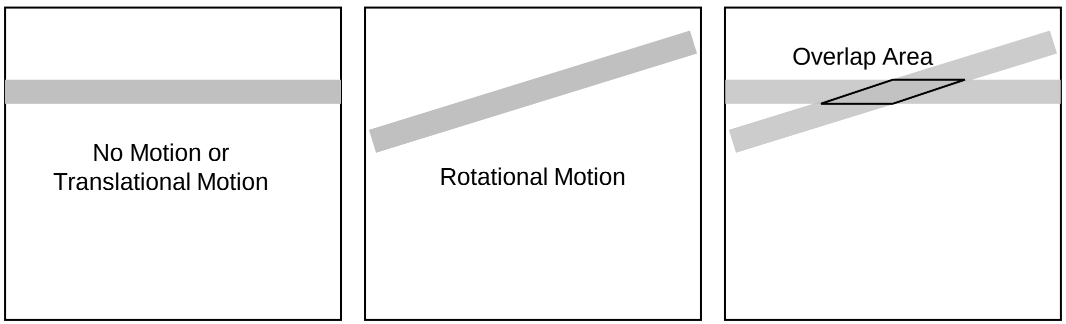
* Regions that were not in original Shepp-Logan (S-L) phantom, ** Slightly modified from original S-L phantom, †3D phantom only, †† Optional region for original S-L phantom, not used herein. Portion subtracted: e.g., **2**[Prop[**1**]] means we subtract an ellipsoid with Ellipsoid 2's geometry (center and dimensions) but Ellipsoid 1's MR properties (relaxation and spin density). Scalp spin density is based on muscle/fat water content since skin water content is highly variable. Tumor spin density is based on its x-ray attenuation coefficient [6].

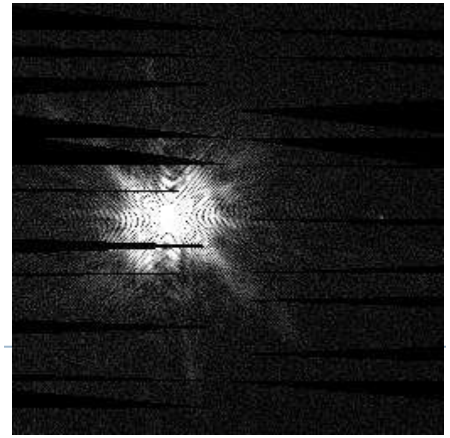
Shepp-Logan Phantom: k-Space

- Using the known Fourier transformation of the Shepp-Logan phantom components (circles and ellipses), one can generate the analytical form of its Fourier transformation
  - Can be sampled arbitrarily to generate uniformly or nonuniformly sampled data for close to real data generation
  - Applications include radial sampling (e.g., CT and MRI), spiral and random sampling (MRI).
- This will be the standard for all evaluation procedures of image reconstruction methods.

Simulation of Medical Image Artifacts

- Motion artifacts in MRI and CT
  - Different parts of k-space correspond to different subject positions
  - Can be simulated using Shepp-Logan phantom





Exercise

- Write a program to verify the projection-slice theorem using a simple 2D phantom (e.g., a basic shape like a square).
- Perform interlaced sampling on a function of your choice with known analytical Fourier transform and verify the interlaced Fourier transform theorem.
- Write a Matlab program to implement the analytical Shepp-Logan phantom and test it using sampling on a uniform grid.