Medical Image Reconstruction
Term II – 2010

Topic 2: Reconstruction from Nonuniformly Sampled k-Space (2)

Professor Yasser Mostafa Kadah

Kadah's Method

Progressive Magnetic Resonance Image Reconstruction Based on Iterative Solution of a Sparse Linear System

Yasser M. Kadah,[1,2] Ahmed S. Fahmy,[2,3] Refaat E. Gabr,[2,3] Keith Heberlein,[1] and Xiaoping P. Hu[1]

- Algebraic Solution
- Iterative reconstruction method that provides an optimal solution in the least-squares sense
- Based on a practical imaging model
- Progressive reconstruction capability
- Simple mechanism to control trade-off between accuracy and speed
- Embedded inhomogeneity correction and spatial domain constraints

Disadvantages of Previous Methods

- Reconstructed images do not represent optimality in any sense
- Variation of performance with form of k-space trajectory
- Lack of explicit methodology to trade-off accuracy and speed of reconstruction
- Not possible to progressively improve the accuracy of reconstruction
- Not possible to embed field inhomogeneity correction or constraints into the reconstruction

Theory

- Assume a piecewise constant spatial domain representing display using pixels
  - Image composed of pixel each of uniform intensity
  - Image can be represented by a sum of 2D RECT functions
- Assume spatial domain to be compact
  - Field of view is always finite in length
- The image can be expressed in terms of gate functions as,

$$f(x,y) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \alpha_{n,m} \cdot \Pi(x - x_n, y - y_m)$$

Theory

- Applying continuous Fourier transform,

$$F(k_x,k_y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\alpha_{n,m}\cdot\Pi(x-x_n,y-y_m)\cdot e^{-j2\pi(k_x x+k_y y)}dxdy,$$

- Hence,

$$F(k_x,k_y)=Sinc(w_x k_x)\cdot Sinc(w_y k_y)\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\alpha_{n,m}\cdot e^{-j2\pi(k_x x_n+k_y y_m)}.$$

- This can be expressed in the form of a linear system as

$$\vec{b}=\mathbf{A}\vec{v}$$

  - A matrix is ~$N^2\times N^2$ and complex-valued

Theory

- Observation: A matrix is ~N^{2}xN^2 and complex-valued
  - Solve a16384x16384 linear system to get a 128x128 image
  - Very difficult to solve in practice because of size

$$
\begin{bmatrix}
\dfrac{F(k_x^0, k_y^0)}{\mathrm{Sinc}\,(w_x k_x^0)\cdot \mathrm{Sinc}\,(w_y k_y^0)} \\
\dfrac{F(k_x^1, k_y^1)}{\mathrm{Sinc}\,(w_x k_x^1)\cdot \mathrm{Sinc}\,(w_y k_y^1)} \\
\vdots \\
\dfrac{F(k_x^{L-1}, k_y^{L-1})}{\mathrm{Sinc}\,(w_x k_x^{L-1})\cdot \mathrm{Sinc}\,(w_y k_y^{L-1})}
\end{bmatrix}_{L\times 1}
= \vec{b} = A\cdot\vec{v}
$$

$$
= \begin{bmatrix}
e^{-j\pi(k_x^0\cdot x_0 + k_y^0\cdot y_0)} & e^{-j\pi(k_x^0\cdot x_0 + k_y^0\cdot y_1)} & \cdots & e^{-j\pi(k_x^0\cdot x_{N-1} + k_y^0\cdot y_{M-1})} \\
e^{-j\pi(k_x^1\cdot x_0 + k_y^1\cdot y_0)} & e^{-j\pi(k_x^1\cdot x_0 + k_y^1\cdot y_1)} & \cdots & e^{-j\pi(k_x^1\cdot x_{N-1} + k_y^1\cdot y_{M-1})} \\
\vdots & \vdots & \cdots & \vdots \\
e^{-j\pi(k_x^{L-1}\cdot x_0 + k_y^{L-1}\cdot y_0)} & e^{-j\pi(k_x^{L-1}\cdot x_0 + k_y^{L-1}\cdot y_1)} & \cdots & e^{-j\pi(k_x^{L-1}\cdot x_{N-1} + k_y^{L-1}\cdot y_{M-1})}
\end{bmatrix}_{L\times N\cdot M}
\cdot
\begin{bmatrix}
\alpha_{0,0} \\ \alpha_{0,1} \\ \vdots \\ \alpha_{N-1,M-1}
\end{bmatrix}_{N\cdot M\times 1}.
$$

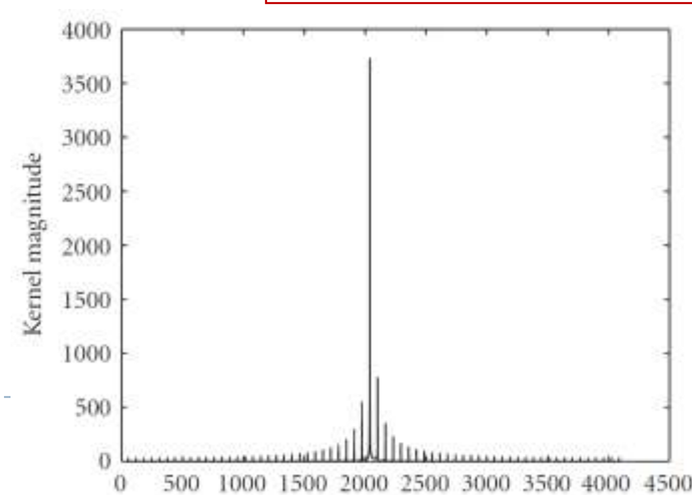
Idea

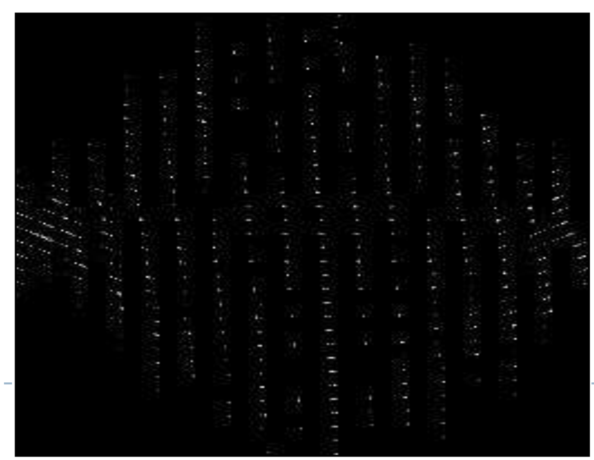
- Problem: A matrix is dense and computational complexity of solution is prohibitive
- Solution Strategy: Try to make the A matrix sparse by seeking a compact representation of rows in terms of suitable basis functions
- Observation: applying a 1-D Fourier transformation to the rows of A matrix results in energy concentration in only a few elements

Methods

- Multiply the rows of the system matrix by the NxM-point discrete Fourier transform matrix H in the following form:

$$H = \frac{1}{\sqrt{NM}} \times \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi/NM} & \cdots & e^{-j2\pi(NM-1)/NM} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & e^{-j2\pi(NM-1)/NM} & \cdots & e^{-j2\pi(NM-1)^2/NM} \end{bmatrix}_{N\cdot M\times N\cdot M}.$$





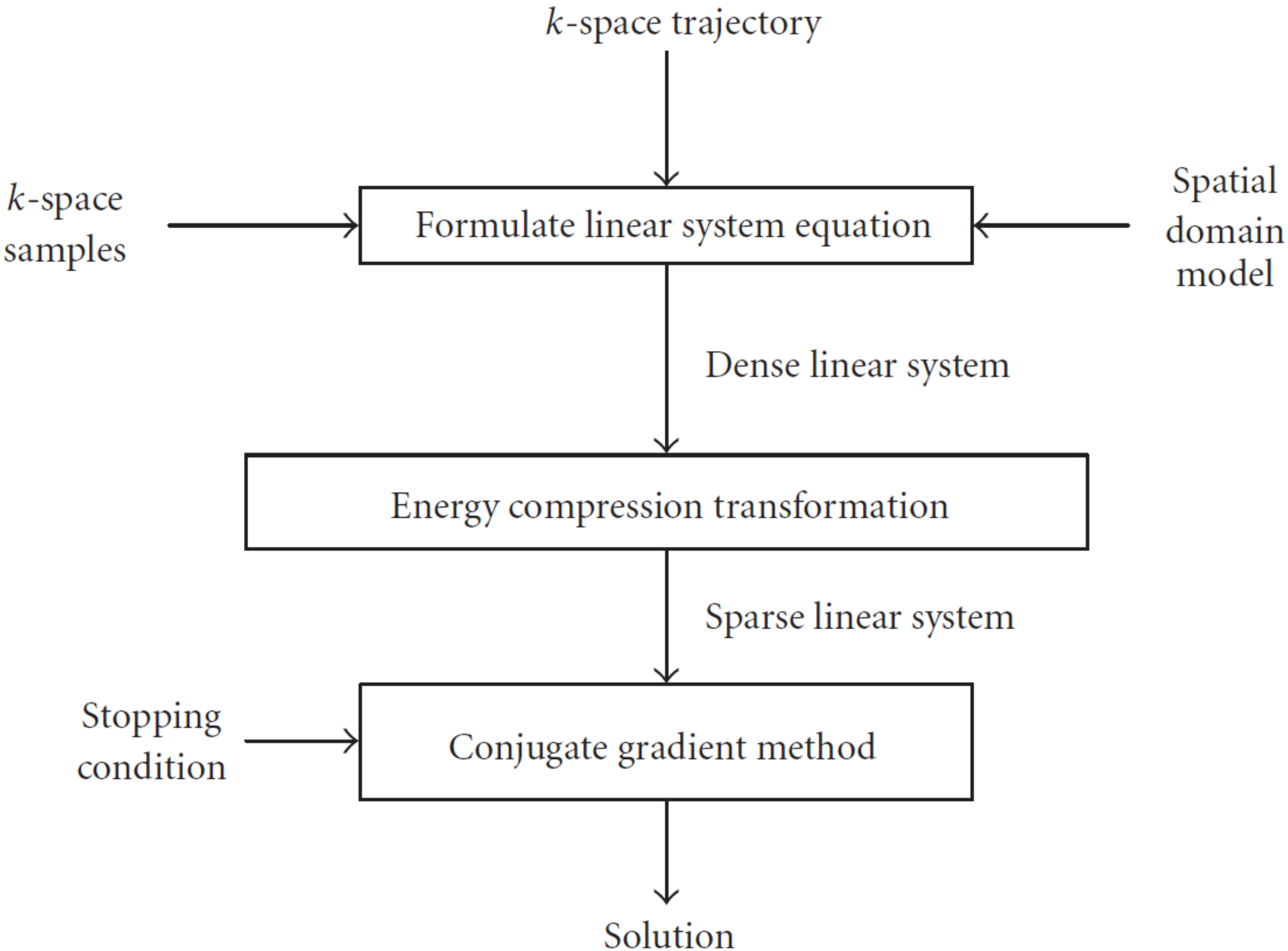
Methods

- How to multiply H without changing the linear system?
  - Row energy compacting transformation converts the system into a sparse linear system as follows:

$$\vec{b} = \mathbf{A}\vec{v} = \mathbf{A} \cdot \mathbf{H}^H \cdot \mathbf{H} \cdot \vec{v} = (\mathbf{H} \cdot \mathbf{A}^H)^H \cdot \vec{V} = \mathbf{M} \cdot \vec{V},$$

- To convert to sparse form, only a percentage η of kernel energy in each row is retained
  - The only parameter in the new method
  - Correlates directly to both image quality and computational complexity
- Sparse matrix techniques are used to store and manipulate the new linear system
  - Since the linear system is sparse, iterative methods such as conjugate gradient can be used to solve the system with very low complexity

Methods


k-space trajectory
k-space samples
Formulate linear system equation
Spatial domain model
Dense linear system
Energy compression transformation
Sparse linear system
Stopping condition
Conjugate gradient method
Solution

Results

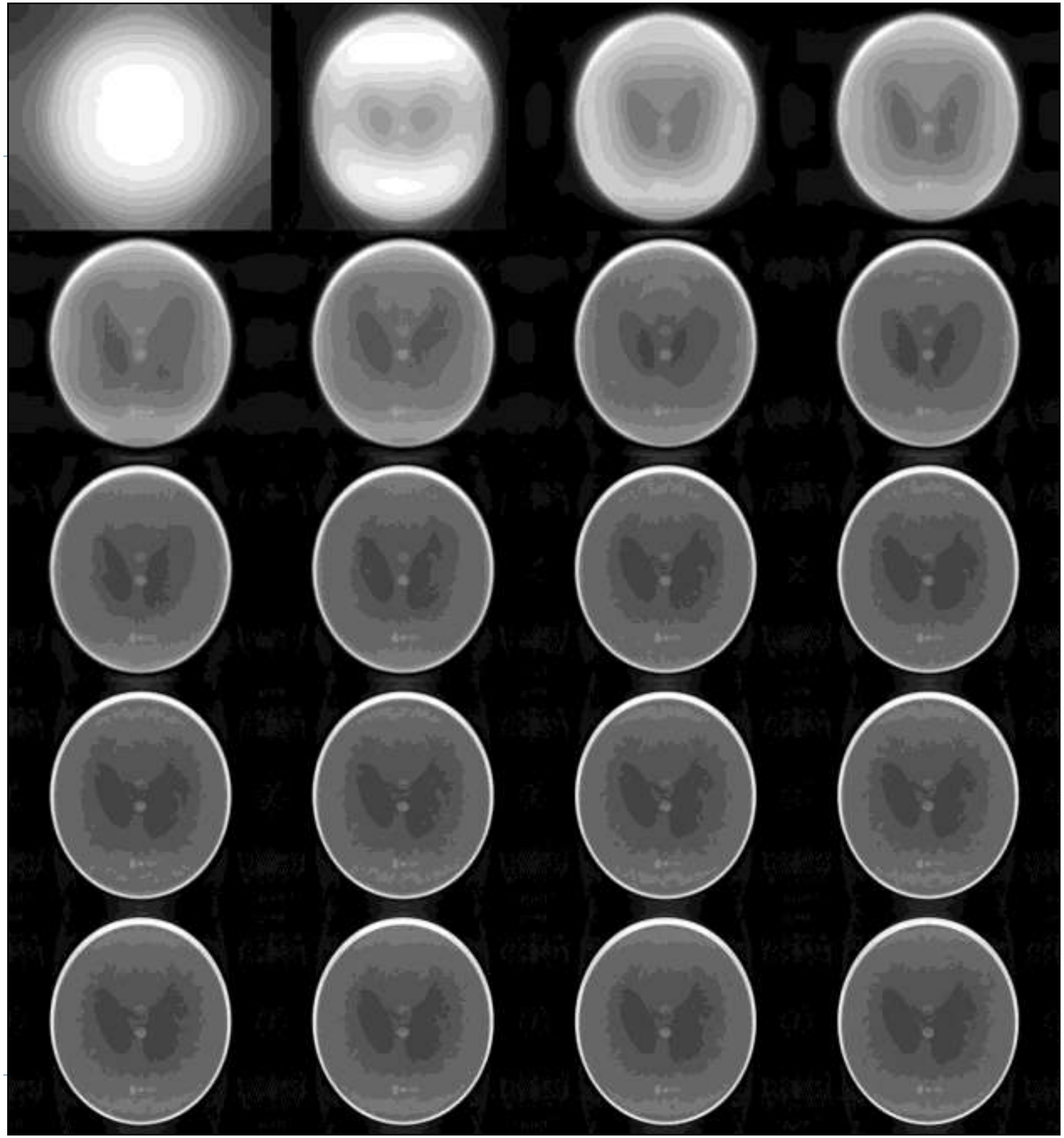
- 256x256

Analytical

Shepp-Logan

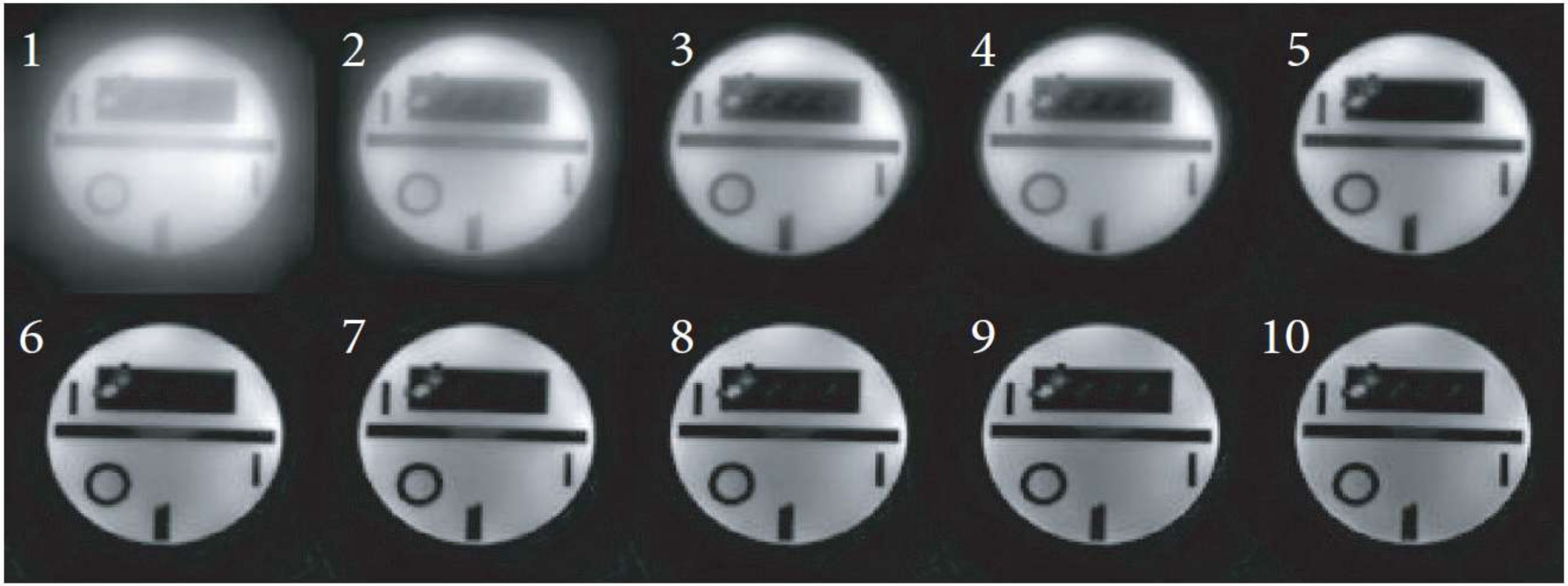
Phantom

(Radial sampling)

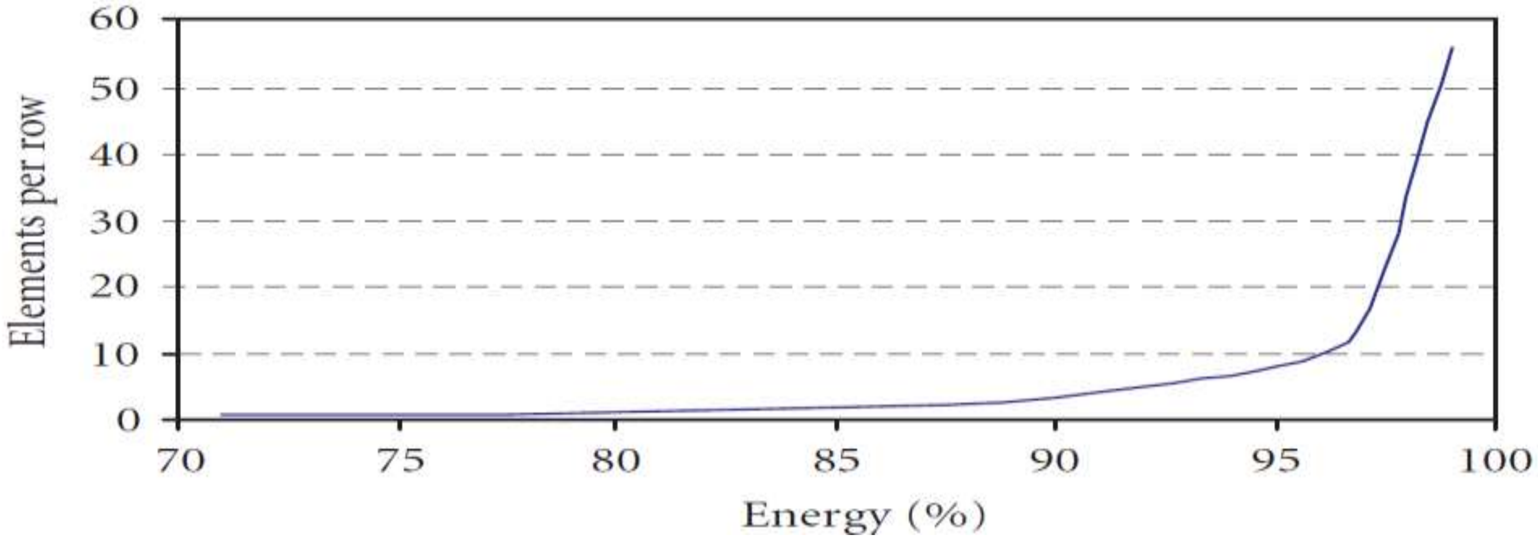


Results

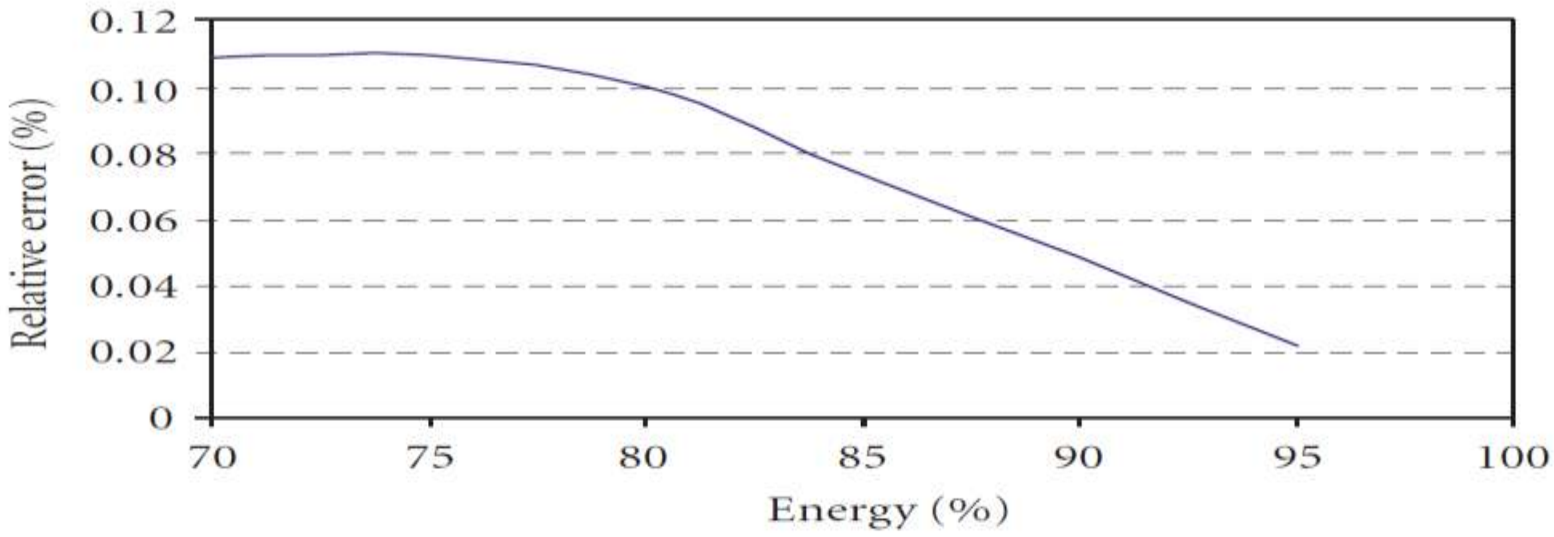
- 256x256 Real data from a resolution phantom at 3T from a Siemens Magnetom Trio system using a spiral trajectory



Results



Elements per row
60
50
40
30
20
10
0
70
75
80
85
90
95
100
Energy (%)



Relative error (%)
0.12
0.10
0.08
0.06
0.04
0.02
0
70
75
80
85
90
95
100
Energy (%)

Discussion

- Full control over the accuracy versus complexity trade-off through η selection
- Computational complexity is comparable to conventional gridding with small kernel
  - **O**$(g(\eta)\cdot L)$ per CGM step, where $g(\eta)$ is the average # of elements/row, L=# of acquired k-space samples
  - Average 4.9 elements/row to retain 92% of energy
- Progressive reconstruction is possible
  - Add more iterations to process
  - Use a different reconstruction table with higher η

Exercise

- Verify the energy compactness transformation and generate Figure 2 (c) for any trajectory you prefer. [1 Point]
- Assuming that we have a rectilinear sampling instead of the nonuniform sampling in this paper, how do you expect the linear system to look like? [1 Points]
- Assume that we are constructing an NxN image, compute the exact number of computation (not an order or computation) detailing the list of computations in each step in the implementation. [1 Point]