Intermediate Physics for Medicine and Biology - Chapter 2

Professor Yasser M. Kadah

Web: http://ymk.k-space.org/courses.htm

Exponential Growth

- An exponential growth process is one in which the rate of increase of a quantity is proportional to that quantity
- Example:

Savings account

$$y_t = y_0(1+b)^t$$

TABLE 2.1. Growth of a savings account earning 5% interest compounded annually, when the initial investment is $100.

Year	Amount	Year	Amount	Year	Amount
1	$105.00	10	$162.88	100	$13,150.13
2	110.25	20	265.33	200	1,729,258.09
3	115.76	30	432.19	300	2.27×10^{8}
4	121.55	40	704.00	400	2.99×10^{10}
5	127.63	50	1146.74	500	3.93×10^{12}
6	134.01	60	1867.92	600	5.17×10^{14}
7	140.71	70	3042.64	700	6.80×10^{16}
8	147.75	80	4956.14	800	8.94×10^{18}
9	155.13	90	8073.04	900	1.18×10^{21}

Exponential Growth

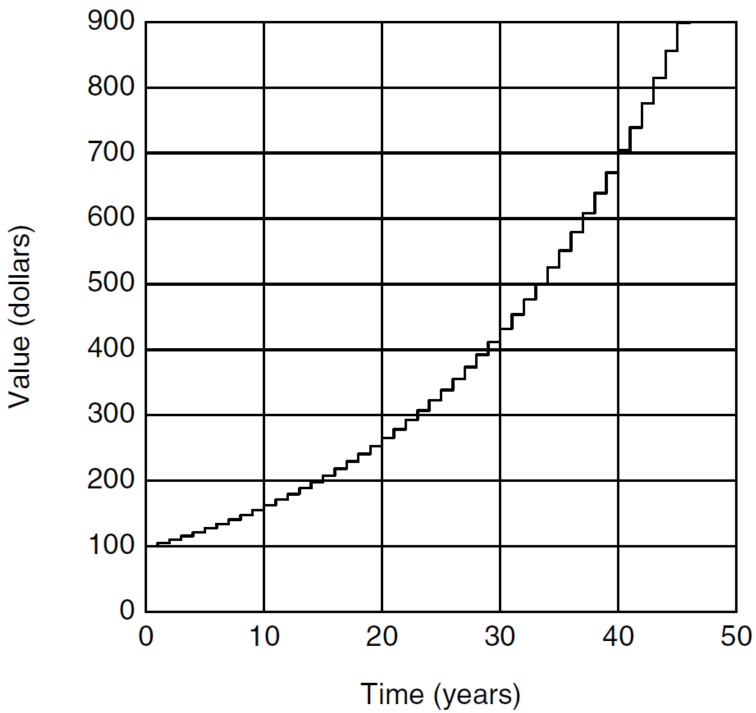


FIGURE 2.1. The amount in a savings account after t years, when the amount is compounded annually at 5% interest.

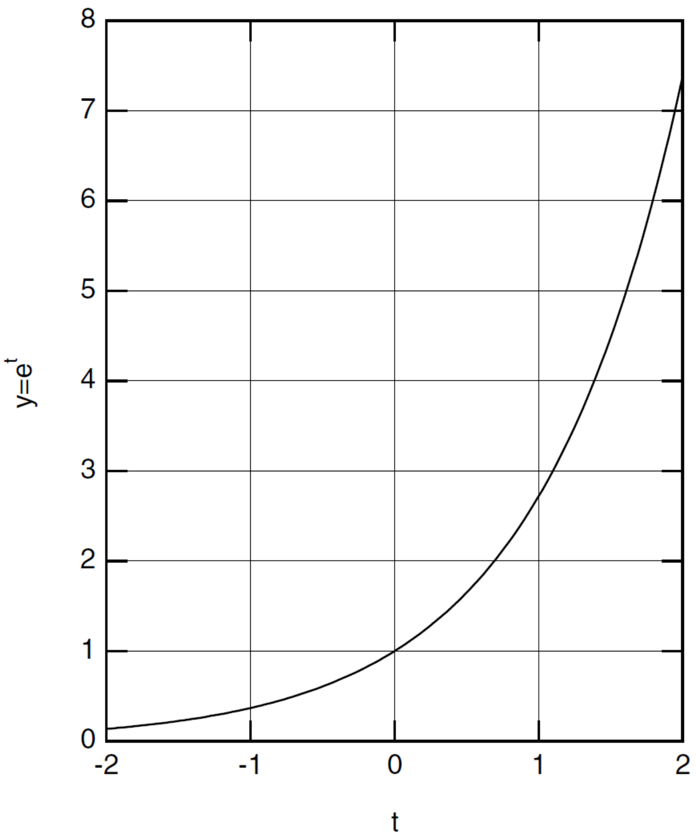


FIGURE 2.2. A graph of the exponential function $y = e^t$.

Exponential Growth: Compounding

- N times/year

$$y_t = y_0 \left(1 + \frac{b}{N}\right)^{Nt}$$

TABLE 2.2. Amount of an initial investment of $100 at 5% annual interest, with different methods of compounding.

Month	Annual	Semiannual	Quarterly	Monthly	Instant
0	$100.00	$100.00	$100.00	$100.000	$100.000
1	100.00	100.00	100.00	100.417	100.418
2	100.00	100.00	100.00	100.835	100.837
3	100.00	100.00	101.25	101.255	101.258
4	100.00	100.00	101.25	101.677	101.681
5	100.00	100.00	101.25	102.101	102.105
6	100.00	102.50	102.52	102.526	102.532
7	100.00	102.50	102.52	102.953	102.960
8	100.00	102.50	102.52	103.382	103.390
9	100.00	102.50	103.80	103.813	103.821
10	100.00	102.50	103.80	104.246	104.255
11	100.00	102.50	103.80	104.680	104.690
12	105.00	105.06	105.09	105.116	105.127

Exponential Growth

- As compounding N $\rightarrow\infty$,

$$y_t = \lim_{N\to\infty}\left\{ y_0\left(1+\frac{b}{N}\right)^{Nt}\right\}$$

$$= \lim_{N\to\infty}\left\{ y_0\left[\left(1+\frac{b}{N}\right)^{N}\right]^{t}\right\}$$

$$= y_0 e^{bt}$$

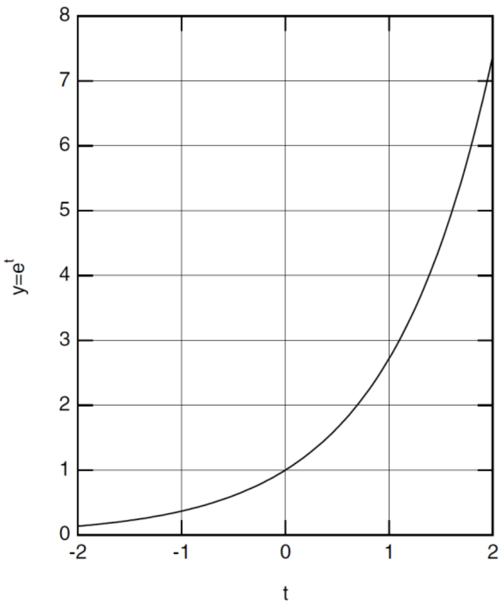


FIGURE 2.2. A graph of the exponential function $y = e^t$.

Exponential Growth

- Differential Equation

$$\frac{dy_t}{dt} = \frac{d}{dt}\left\{y_0 e^{bt}\right\} = by_0 e^{bt} = by_t$$

Exponential Decay

- Example: assume b>0

$$y_t = y_0 e^{-bt}$$

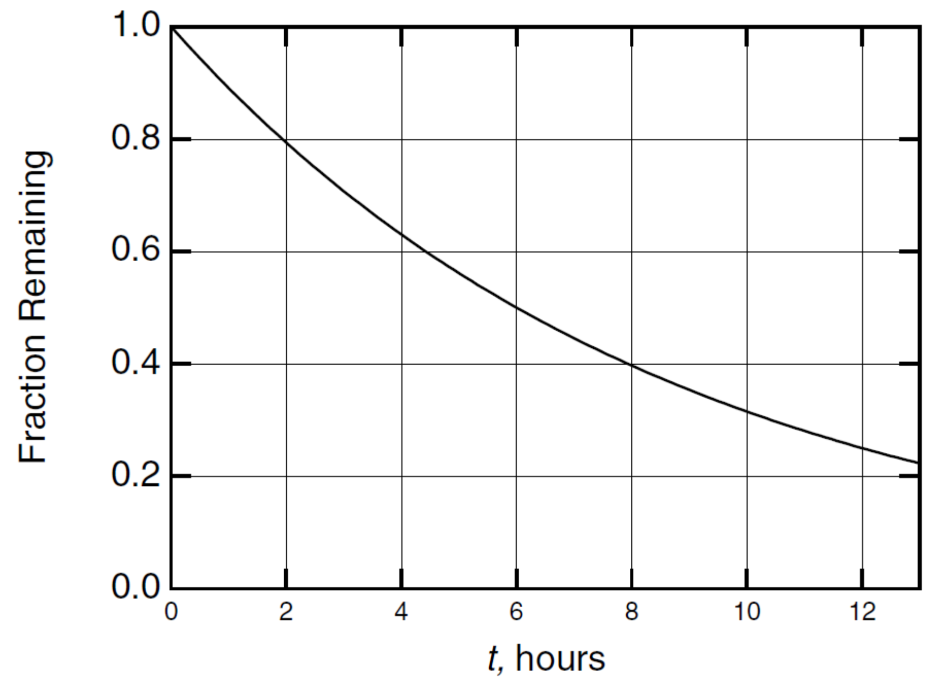


FIGURE 2.3. A plot of the fraction of nuclei of ^{99m}Tc surviving at time t.

Exponential Decay

- Half-Life $T_{½}$: Length of time required for y_t to decrease to ½ its original value

$$y_{T_{1/2}} = 0.5 y_0 \Leftrightarrow e^{-bT_{1/2}} = 0.5$$

$$T_{1/2} \approx \frac{0.693}{b}$$

- Note: Doubling time T_2 is same value

Exponential Decay

- Example: Radioactive decay of ^{99m}Tc
  - Decay rate: $b = 0.1155\ h^{-1}$
  - $T_{½} = 0.693/0.1155 = 6$ h

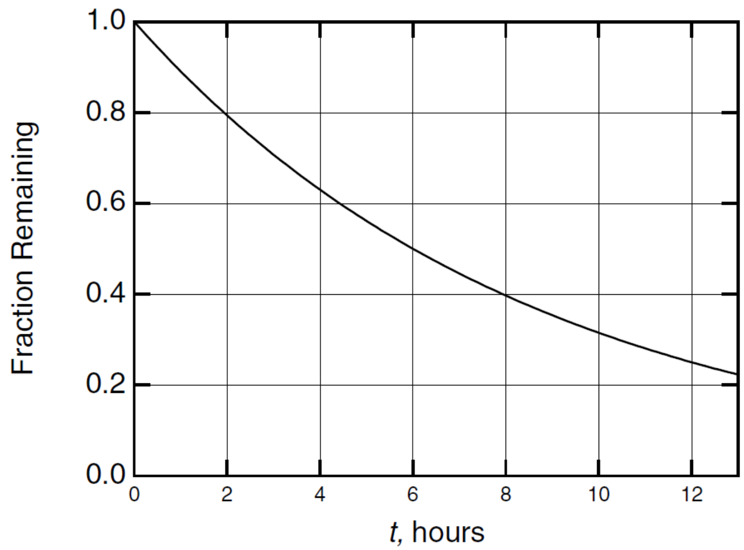


FIGURE 2.3. A plot of the fraction of nuclei of ^{99m}Tc surviving at time t.

Semilog Paper

$$\log y_t = \log y_0 e^{bt} = \log y_0 + bt$$

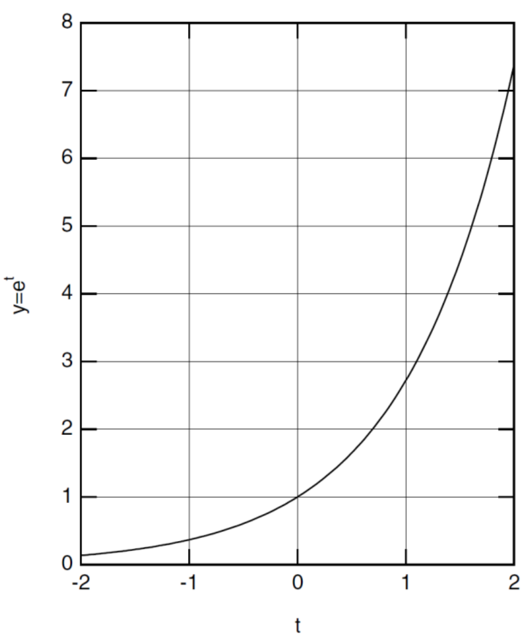


FIGURE 2.2. A graph of the exponential function $y = e^t$.

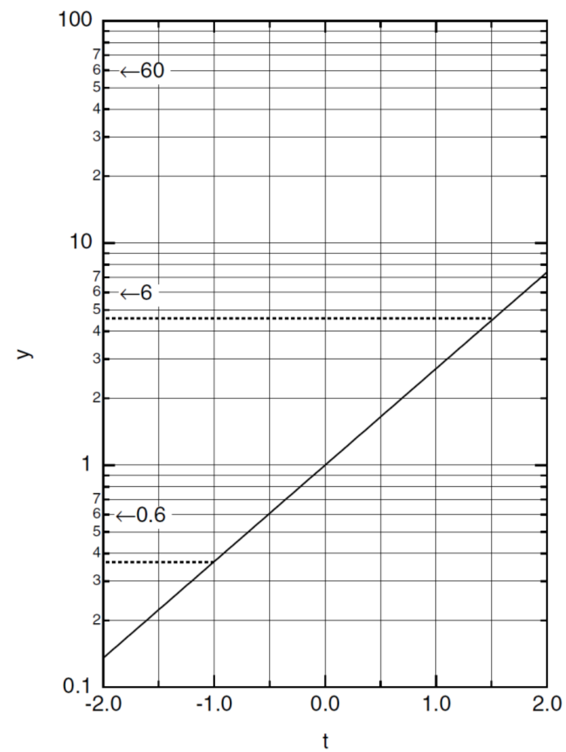


FIGURE 2.4. A plot of the exponential function on semilog paper.

Semilog Paper: Example

$$\log y_t = \log y_0 e^{-bt}$$
$$= \log y_0 - bt$$

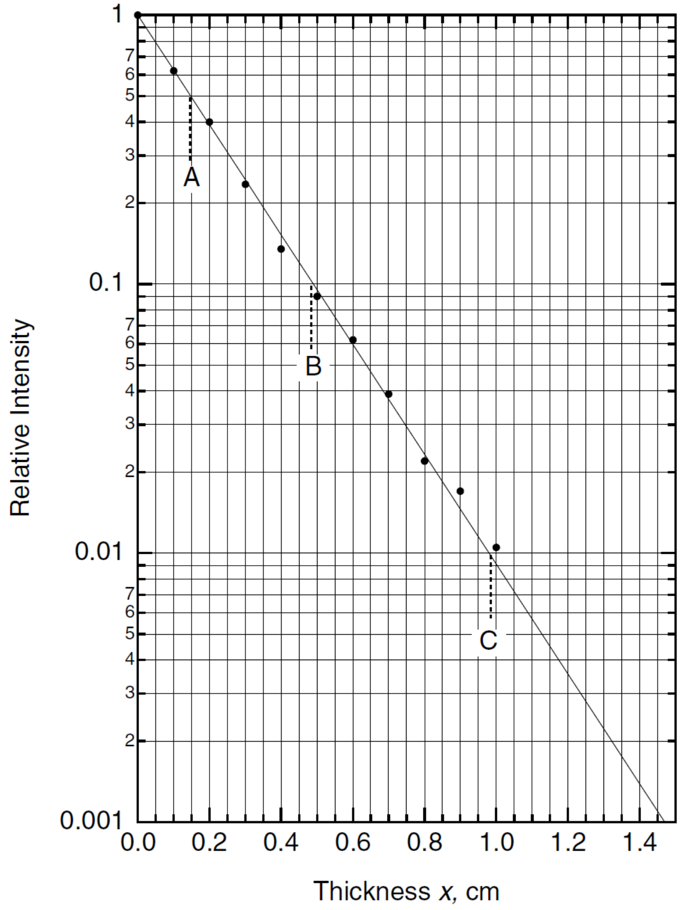


FIGURE 2.5. A semilogarithmic plot of the intensity of light after it has passed through an absorber of thickness x.

Variable Rates

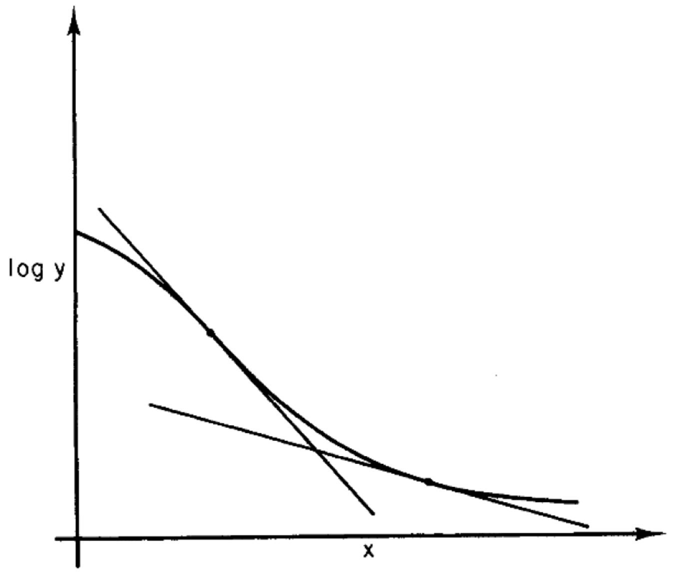


FIGURE 2.6. A semilogarithmic plot of y vs x when the decay rate is not constant. Each tangent line represents the instantaneous decay rate for that value of x.

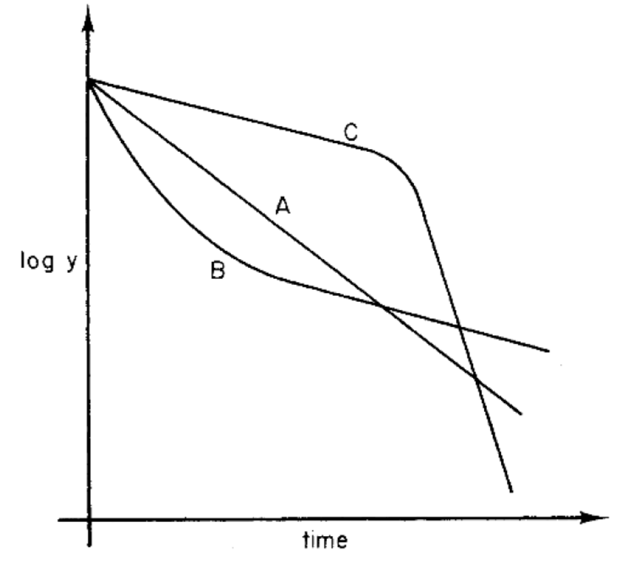


FIGURE 2.7. Semilogarithmic plots of the fraction of a population surviving in three different diseases. The death rates (decay constants) depend on the duration of the disease.

Variable Rates: Example

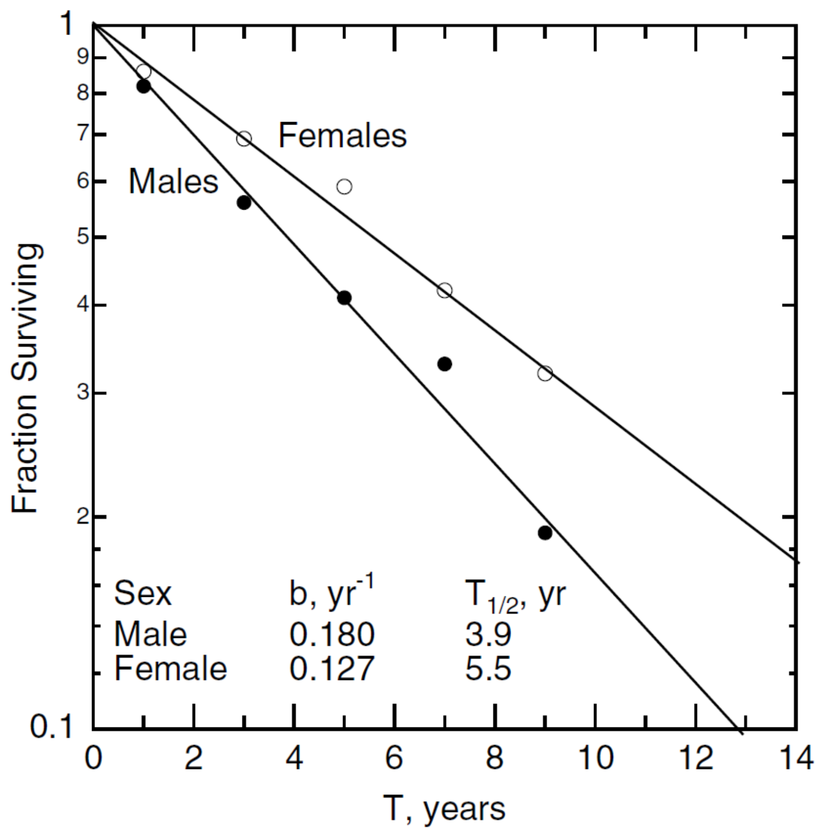
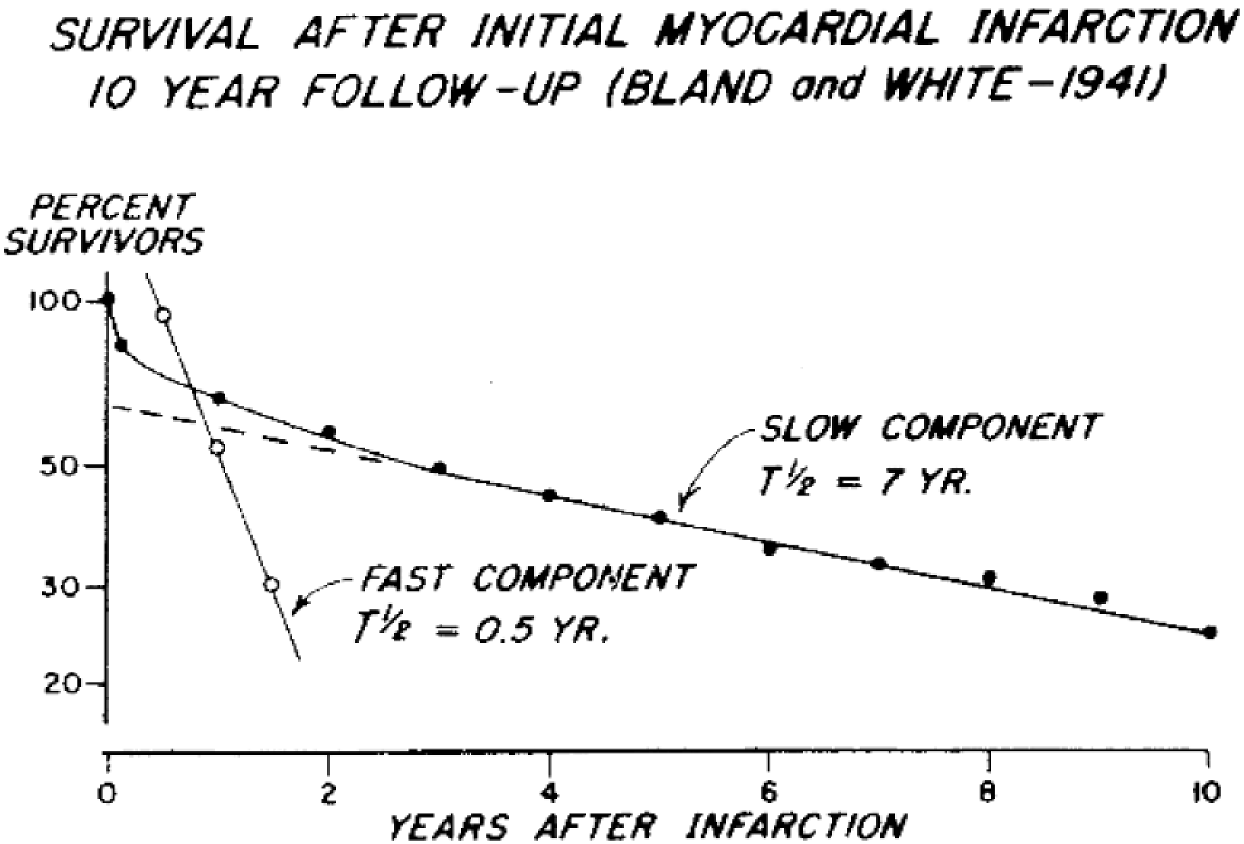


FIGURE 2.8. Survival of patients with congestive heart failure. Data are from McKee *et al.* (1971).

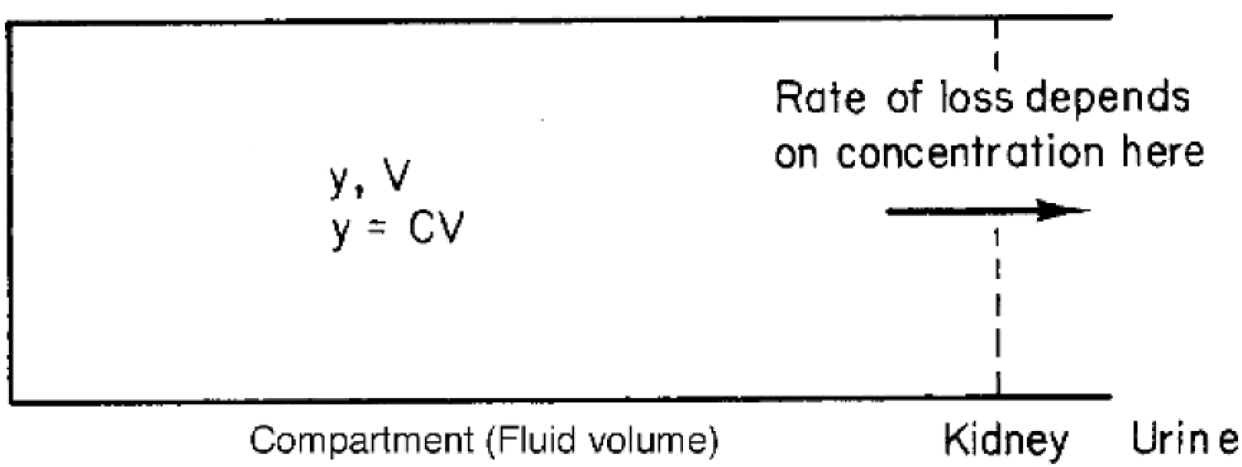
Variable Rates: Example



Clearance

- Clearance K is defined by,

$$\frac{dy}{dt} = -KC = -K\left(\frac{y}{V}\right) = -\left(\frac{K}{V}\right)y$$



Multiple Decay Paths

- Multiple decay processes

$$\frac{dy}{dt} = -b_1 y - b_2 y - b_3 y - \cdots = -(b_1 + b_2 + b_3 + \cdots)\, y = -by$$

- Half-life

$$\frac{0.693}{T} = \frac{0.693}{T_1} + \frac{0.693}{T_2} + \frac{0.693}{T_3} + \cdots \Rightarrow T = (T_1 + T_2 + T_3 + \cdots)$$

Decay Plus Input at a Constant Rate

$$\frac{dy}{dt} = a - by \Rightarrow y = \frac{a}{b}(1 - e^{-bt})$$

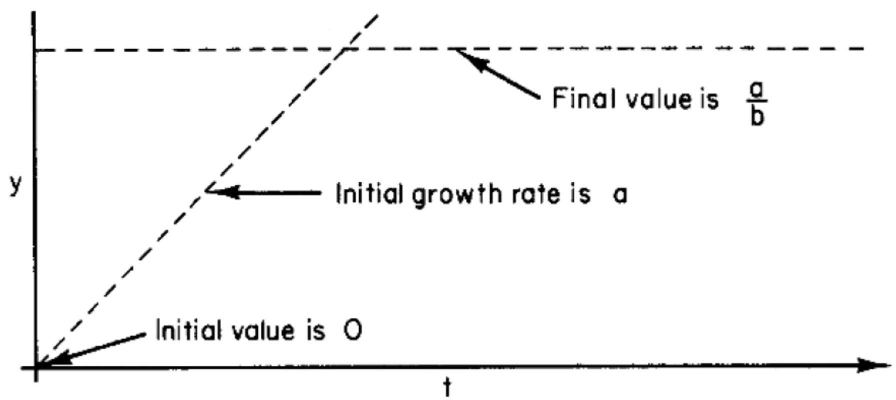


FIGURE 2.11. Sketch of the initial slope a and final value a/b of y when $y(0) = 0$.

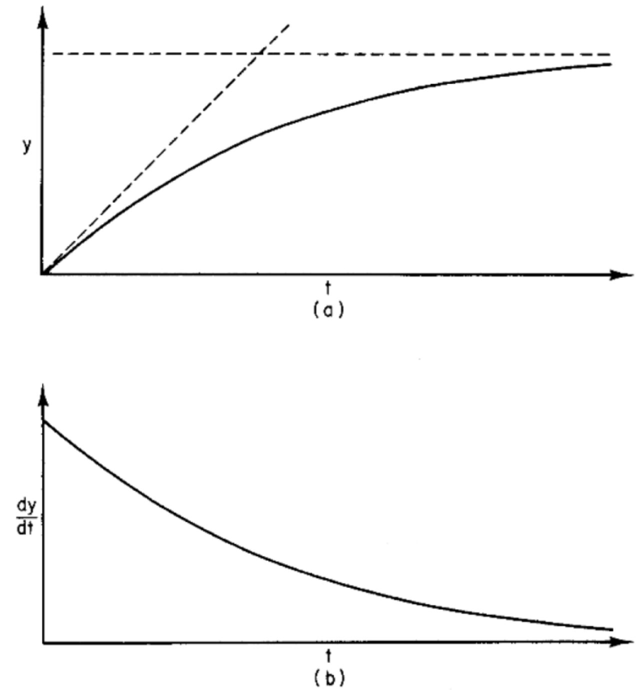


FIGURE 2.12. (a) Plot of $y(t)$. (b) Plot of dy/dt.

Decay with Multiple Half-Lives: Fitting Exponentials

$$y = y_1 + y_2$$
$$= A_1 e^{-b_1 t} + A_2 e^{-b_2 t}$$

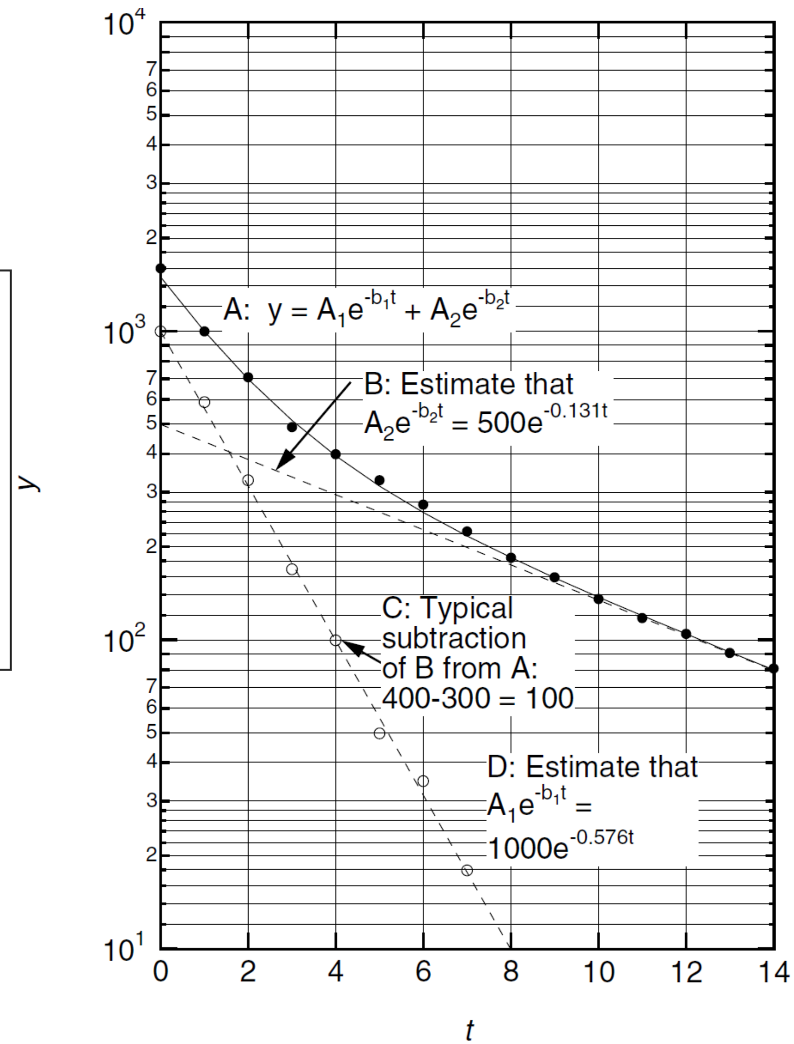


FIGURE 2.13. Fitting a curve with two exponentials.

Log-Log Plots

$$y = Bx^n$$

$$\log y = \log B + n \log x$$

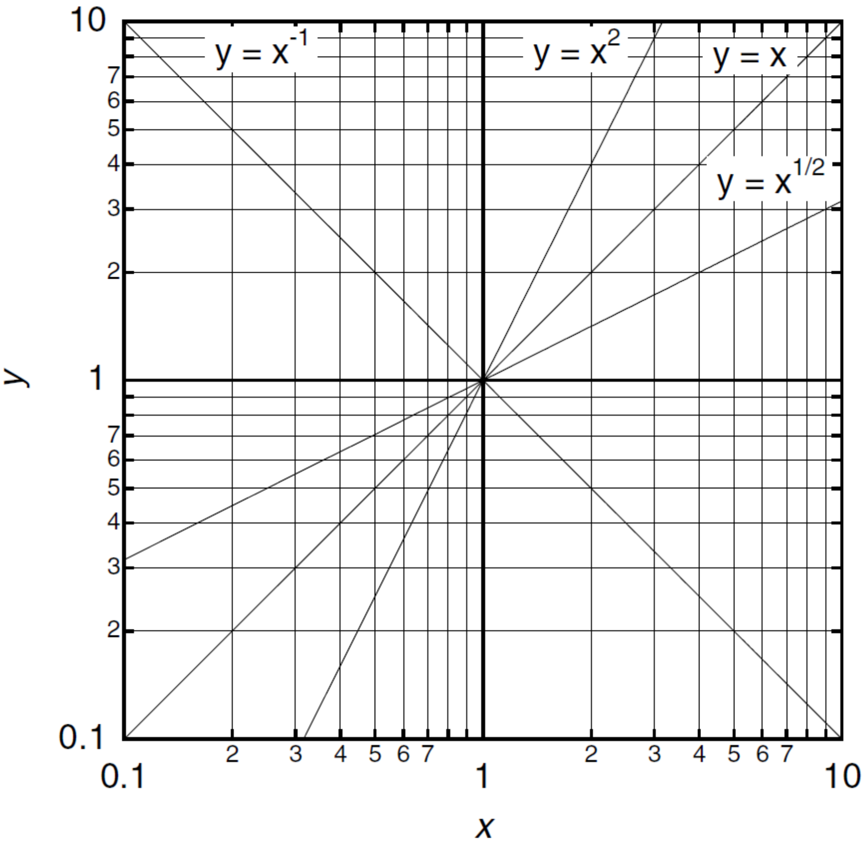
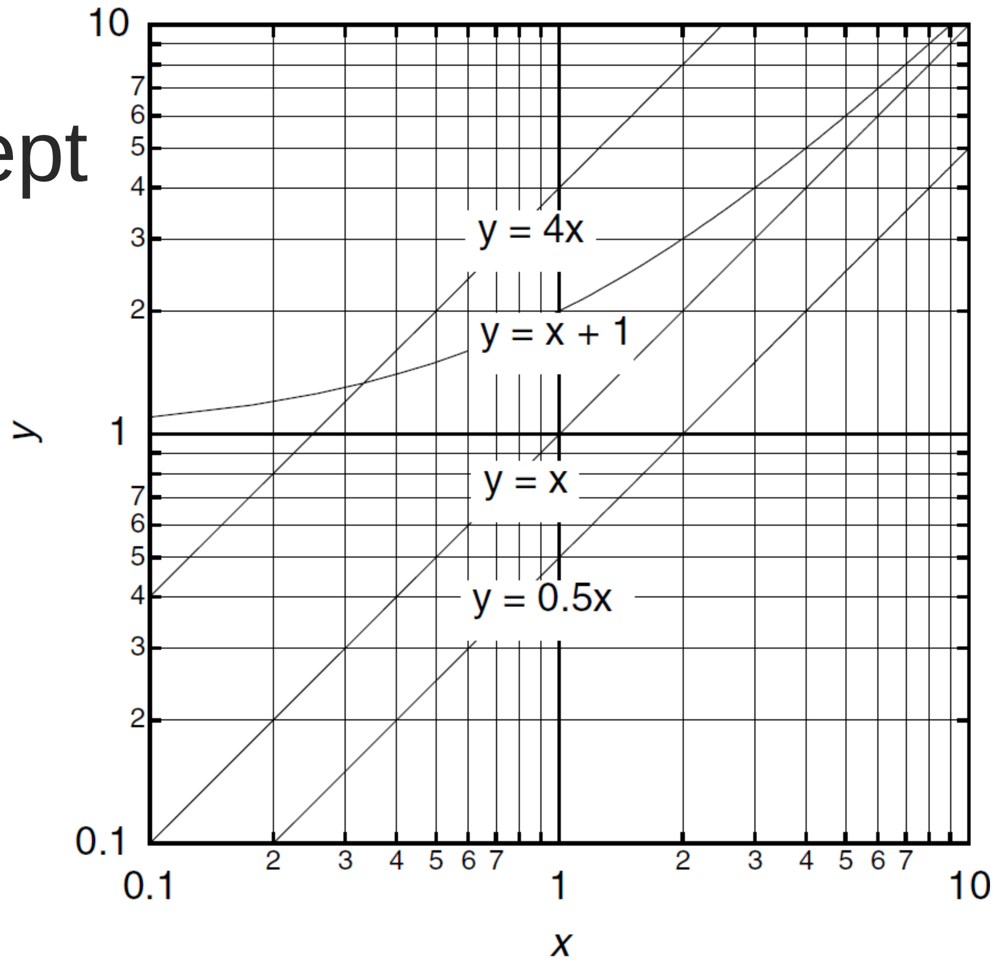


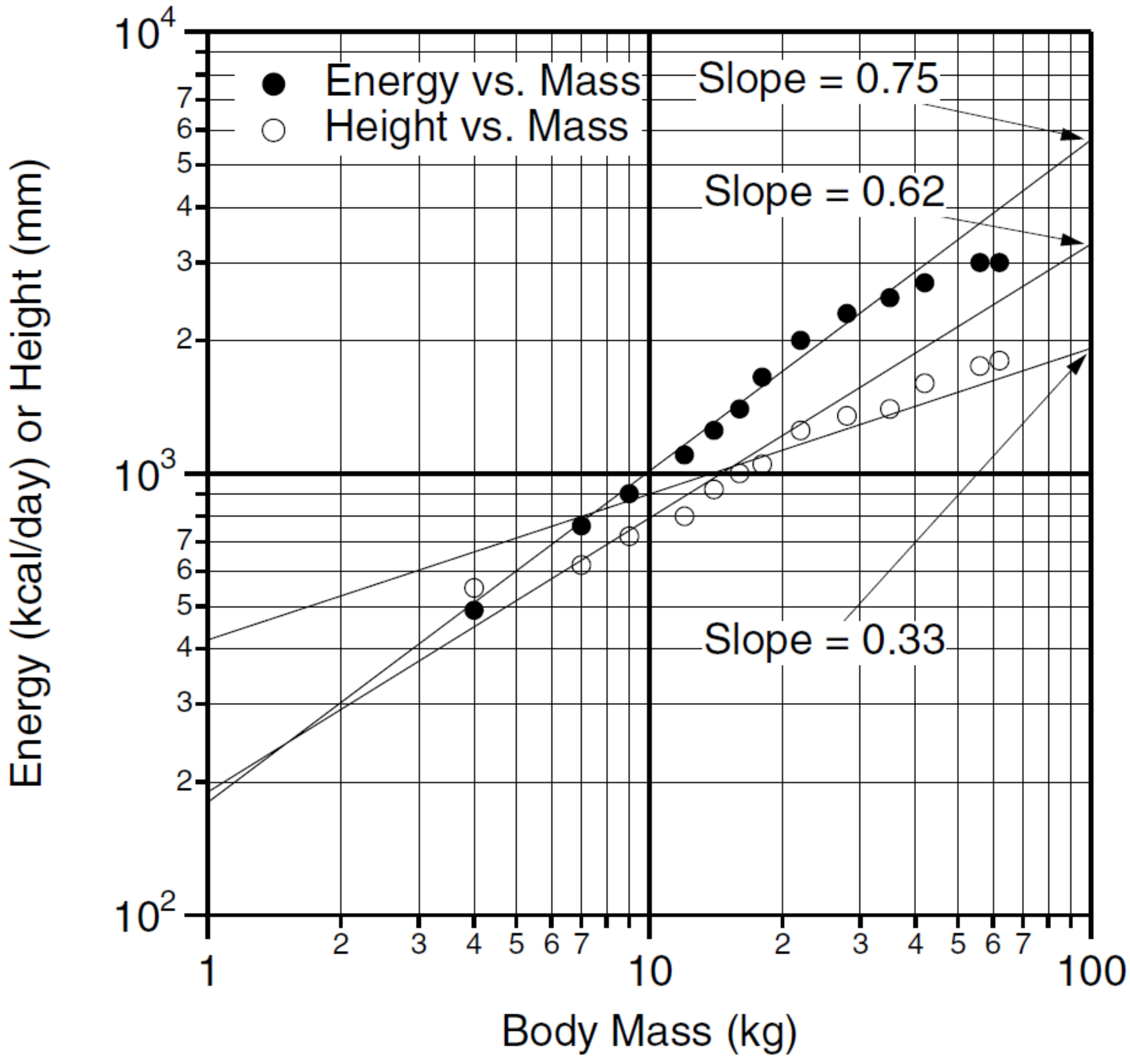
FIGURE 2.15. Log-log plots of $y = x^n$ for different values of n. When $x = 1$, $y = 1$ in every case.

Log-Log Plots

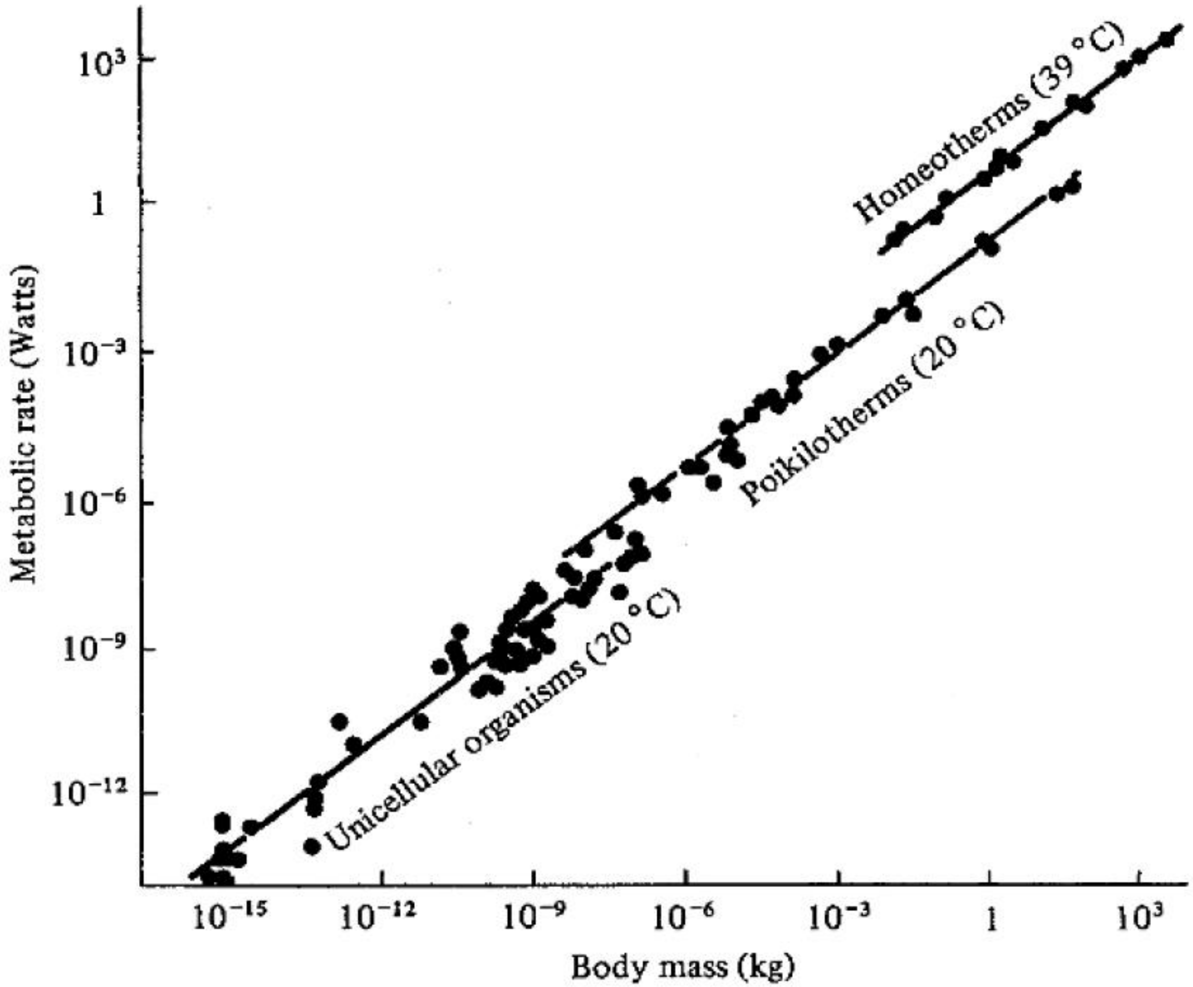
- Scaling
- Nonzero intercept



Example: Food Consumption



Example: Basal Metabolic Rate



Problem Assignment

- Posted on class web site

Web: http://ymk.k-space.org/courses.htm