Intermediate Physics for Medicine and Biology - Chapter 4

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Transport in an Infinite Medium

- Definitions
- Continuity equations
- Brownian motion
- Motion in a gas
- Motion in a liquid
- Diffusion
- Applications

Definitions

- Flow rate, volume *flux* or volume current (*i*)
  - Total volume of material transported per unit time
  - Units: m^3s^{-1}
- Mass *flux*
- Particle *flux*

Definitions

- Particle *fluence*
  - Number of particles transported per unit area across an imaginary surface
  - Units: m^{-2}
- Volume *fluence*
  - Volume transported per unit area across an imaginary surface
  - Units: $m^3 m^{-2} = m$

Definitions

- Fluence rate or flux density
  - Amount of "something" transported across an imaginary surface per unit area per unit time
  - Vector pointing in the direction the "something" moves and is denoted by **j**
  - Units: "something" $m^{-2}s^{-1}$
  - Subscript to denote what "something" is

Definitions

TABLE 4.1. Units and names for j and jS in various fields.

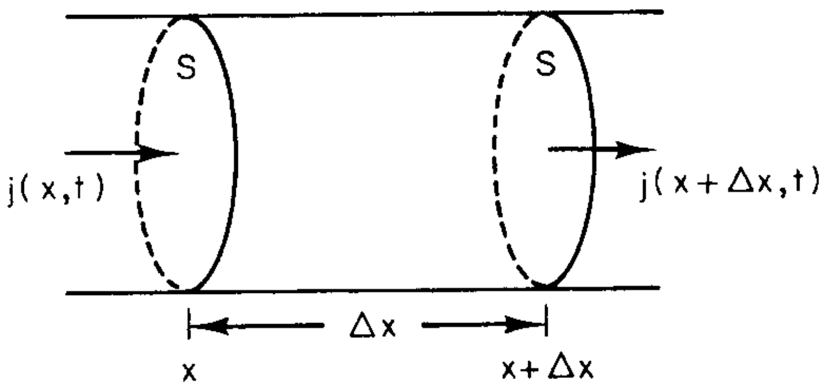
	j		jS	
	Units	Names	Units	Names
Particles	$\mathrm{m}^{-2}\ \mathrm{s}^{-1}$	Particle fluence rate Particle current density Particle flux density Particle flux	s^{-1}	Particle flux Particle current Particle flux
Electric charge	$\mathrm{C\ m}^{-2}\ \mathrm{s}^{-1}$ or $\mathrm{A\ m}^{-2}$	Current density	$\mathrm{C\ s}^{-1}$ or A	Current
Mass	$\mathrm{kg\ m}^{-2}\ \mathrm{s}^{-1}$	Mass fluence rate Mass flux density Mass flux	$\mathrm{kg\ s}^{-1}$	Mass flux Mass flow
Energy	$\mathrm{J\ m}^{-2}\ \mathrm{s}^{-1}$ or $\mathrm{W\ m}^{-2}$	Energy fluence rate Intensity Energy flux	$\mathrm{J\ s}^{-1}$ or W	Energy flux Power

Continuity Equation: 1D

- We deal with substances that do not “appear” or “disappear”
  - Conserved
- Conservation of mass leads to the derivation of the continuity equation

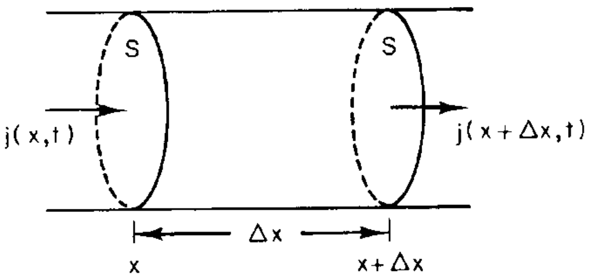
Continuity Equation: 1D

- Consider the case of a number of particles
  - Fluence rate: j particles/unit area/unit time
- Value of j may depend on position in tube and time
  - $j = j(x,t)$
- Let volume of paricles in the volume shown to be $N(x,t)$
  - Change after $\Delta t = \Delta \mathrm{N}$



Continuity Equation: 1D

$$\boxed{\Delta N = \left[j(x,t) - j(x+\Delta x, t) \right] S \Delta t}$$



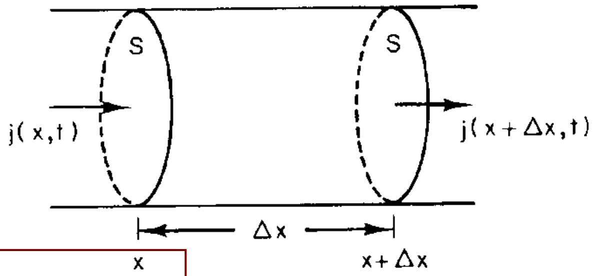
- As $\Delta x \to 0$,

$$\boxed{j(x,t) - j(x+\Delta x, t) = -\frac{\partial j(x,t)}{\partial x}\Delta x}$$

- Similarly, increase in N(x,t) is,

$$\boxed{\Delta N(x,t) = N(x,t+\Delta t) - N(x,t) = \frac{\partial N}{\partial t}\Delta t}$$

Continuity Equation: 1D



- Hence,

$$\boxed{\frac{\partial}{\partial t} N(x,t) = -(S\Delta x)\frac{\partial}{\partial x}\, j(x,t)}$$

- Then, the continuity equation in 1D is,

$$\boxed{\frac{\partial C}{\partial t} = -\frac{\partial j}{\partial x}}$$

Solvent Drag (Drift)

- One simple way that solute particles can move is to drift with constant velocity.
  - Carried along by the solvent,
- Process called *drift or solvent drag.*

$$\boxed{\mathbf{j}_s = C\,\mathbf{j}_v}$$

Brownian Motion

- Application of thermal equilibrium at temperature T
- Kinetic energy in 1D = $k_B T/2$
- Kinetic energy in 3D = $3k_B T/2$
- Random motion $\longrightarrow$ mean velocity $\bar{v} = 0$
  - can only deal with mean-square velocity $\overline{v^2}$

$$\boxed{\frac{1}{2}m\overline{v^2} = \frac{3k_B T}{2} \Rightarrow v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\frac{3k_B T}{m}}}$$

Brownian Motion

TABLE 4.2. Values of the rms velocity for various particles at body temperature.

Particle	Molecular weight	Mass (kg)	v_{rms} (m s^{-1})
H_2	2	3.4×10^{-27}	1940
H_2O	18	3×10^{-26}	652
O_2	32	5.4×10^{-26}	487
Glucose	180	3×10^{-25}	200
Hemoglobin	65 000	1×10^{-22}	11
Bacteriophage	6.2×10^6	1×10^{-20}	1.1
Tobacco mosaic virus	40×10^6	6.7×10^{-20}	0.4
E. coli		2×10^{-15}	0.0025

Motion in a Gas

- Brownian motion of particles: collisions
- Mean Free Path
  - Average distance between successive collisions
- Collision Time
  - Average time between successive collisions

Motion in a Gas

- Consider $N(x)$ to be number of particles without collision after a distance x
- For short distances dx, probability of collision is proprtional to dx

$$\boxed{dN = N(x)\left(\frac{1}{\lambda}\right)dx} \rightarrow \boxed{N(x) = N_0 e^{-x/\lambda}}$$

Motion in a Gas

- Average distance = mean free path

$$\bar{x} = \frac{1}{N_o}\int_0^\infty x\frac{N(x)}{\lambda}dx = -\lambda\left[e^{-x/\lambda}\left(\frac{x}{\lambda}+1\right)\right]_0^\infty = \lambda$$

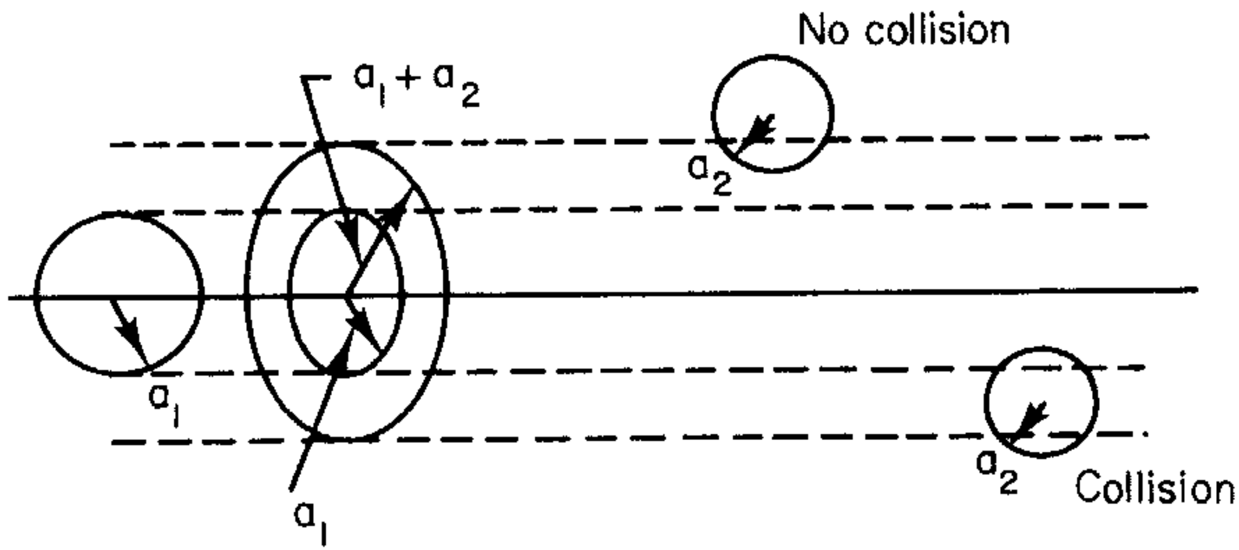
- Similar argument can be made for time

$$N(t) = N_0 e^{-t/t_c}$$

  - Collision time = t_c

Motion in a Gas

- Need to evaluate λ and t_c
- Consider one particle moving with a radius a_1
- Consider stationary particles with radius a_2



- Calculate collision possibilities

Motion in a Gas

- After moving a distance x, volume covered is given by,

$$\boxed{V(x) = \pi(a_1 + a_2)^2 x}$$

- On average, when a particle travels mean free path, there is one collision
  - Average number of particles in V(λ)=1
  - Concentration = 1/V(λ)

$$\boxed{C = 1/V(\lambda) = 1/\pi(a_1 + a_2)^2 \lambda \Rightarrow \lambda = \frac{1}{\pi(a_1 + a_2)^2 C}}$$

Motion in a Gas

- Collision *Cross Section* is $\pi(a_1 + a_2)^2$
  - Important for radiation interaction
- Example: gas at STP, volume of 1 mol = 22.4 L (C= 2.7×10^{25} m^{-3}), a_1=a_2=0.15 nm
  - λ=0.13 μm
  - 1000 times the molecular size
  - Assumption of infrequent collisions justified

Motion in a Gas

- Given mean free path λ,

$$\boxed{t_c = \frac{\lambda}{\bar{v}}}$$

- Taking the average speed as v_{rms},

$$\boxed{t_c \approx \lambda \left(\frac{m}{3k_B T} \right)^{1/2}}$$

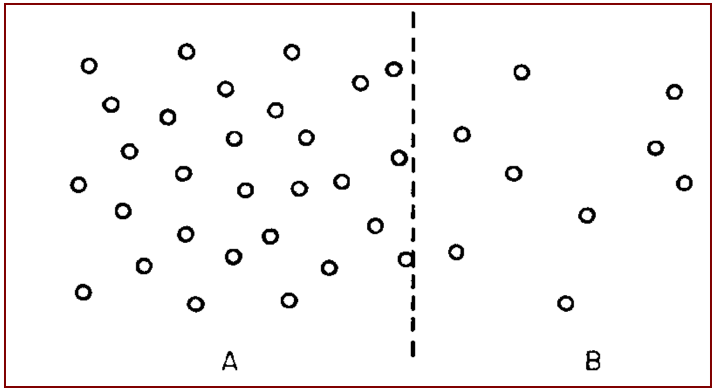
  - Dependence on $m^{1/2}$ and λ
  - For air and room temperature, $t_c = 2\times10^{-10}$s

Motion in a Liquid

- Direct substitution in Gas equations?
- For water,
  - λ=a=0.1 nm $\rightarrow$ assumption broken
  - $t_c \sim 10^{-13}$ s $\rightarrow$ much more frequent
  - Wrong calculations
  - However, concept appears to be valid!

Diffusion: Fick's First Law

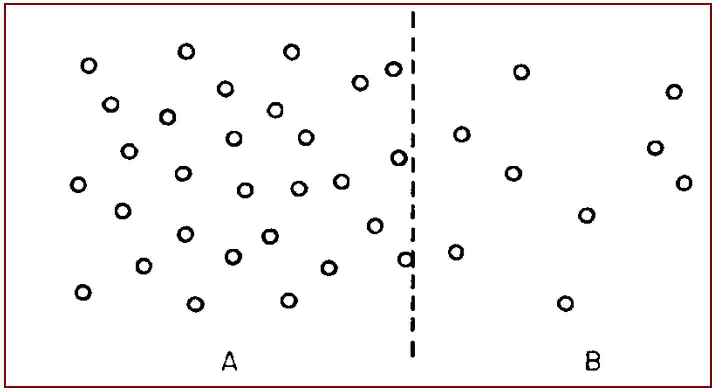
- Diffusion: random movement of particles from a region of higher concentration to a region of lower concentration
- Diffusing particles move independently
- Solvent at rest
  - Solute transport



Diffusion: Fick's First Law

- If solute concentration is uniform, no net flow
- If solute concentration is different, net flow occurs

$$j_x = -D \frac{\partial C}{\partial x}$$



D: Diffusion constant (m^2s^{-1})

Diffusion: Fick's First Law

TABLE 4.3. Various forms of the transport equation.

Substance flowing	Equation	Units of j	Units of the constant
Particles	$j_s = -D \frac{\partial C}{\partial x}$	$m^{-2}\ s^{-1}$	$m^2\ s^{-1}$
Mass	$j_m = -D \frac{\partial C}{\partial x}$	$kg\ m^{-2}\ s^{-1}$	$m^2\ s^{-1}$
Heat	$j_H = -\kappa \frac{\partial T}{\partial x}$	$J\ m^{-2}\ s^{-1}$ or $kg\ s^{-3}$	$J\ K^{-1}\ m^{-1}\ s^{-1}$
Electric charge	$j_e = -\sigma \frac{\partial V}{\partial x}$	$C\ m^{-2}\ s^{-1}$	$C\ m^{-1}\ s^{-1}\ V^{-1}$ or $\Omega^{-1}\ m^{-1}$
Viscosity (y component of momentum transported in the x direction)	$\frac{F}{S} = -\eta \frac{\partial v_y}{\partial x}$	$N\ m^{-2}$ or $kg\ m^{-1}\ s^{-2}$	$kg\ m^{-1}\ s^{-1}$ or Pa s

Diffusion: Fick's Second Law

- Consider 1D case
- Fick's first law

$$\boxed{j_x = -D\frac{\partial C}{\partial x}}$$

- Continuity equation

$$\boxed{\frac{\partial C}{\partial t} = -\frac{\partial j}{\partial x}}$$

Diffusion: Fick's Second Law

- Combining Fick's first law and continuity equation,

$$\boxed{\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}}$$

→Fick's second law

- 3D Case,

$$\boxed{\frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right)}$$

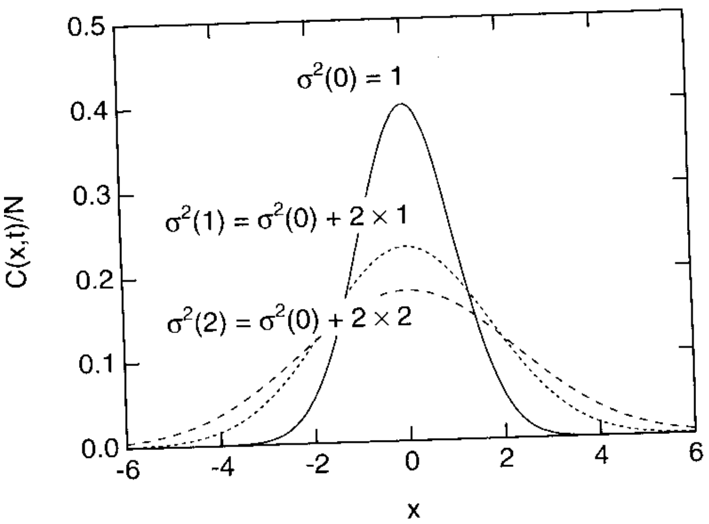
Diffusion: Fick's Second Law

- Solving Fick's second law for C(x,t)
  - Substitution

$$\boxed{C(x,t) = \frac{N}{\sqrt{2\pi}\sigma(t)} e^{-x^2/2\sigma^2(t)}}$$

where,

$$\boxed{\sigma^2(t) = 2Dt + \sigma^2(0)}$$



Applications

- Kidney dialysis
- Tissue perfusion
- Blood oxygenation in the lung

Problem Assignments

- Information posted on web site