Signals and Systems – Chapter 0

Mathematical Preliminaries

Prof. Yasser Mostafa Kadah

Overview of Chapter 0

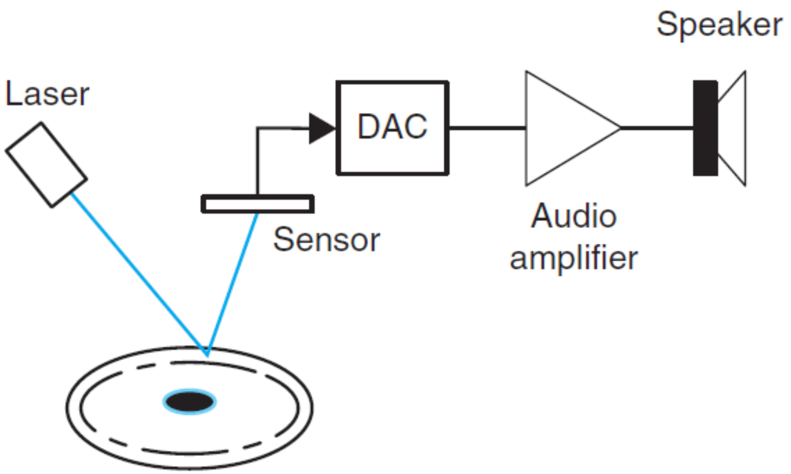
- Importance of the theory of signals and systems
- Mathematical preliminaries
- Matlab introduction (covered in section)

Introduction

- Learning how to represent signals in analog as well as in digital forms and how to model and design systems capable of dealing with different types of signals
- Most signals come in analog form
- Trend has been toward digital representation and processing of data
  - Computer capabilities increase continuously

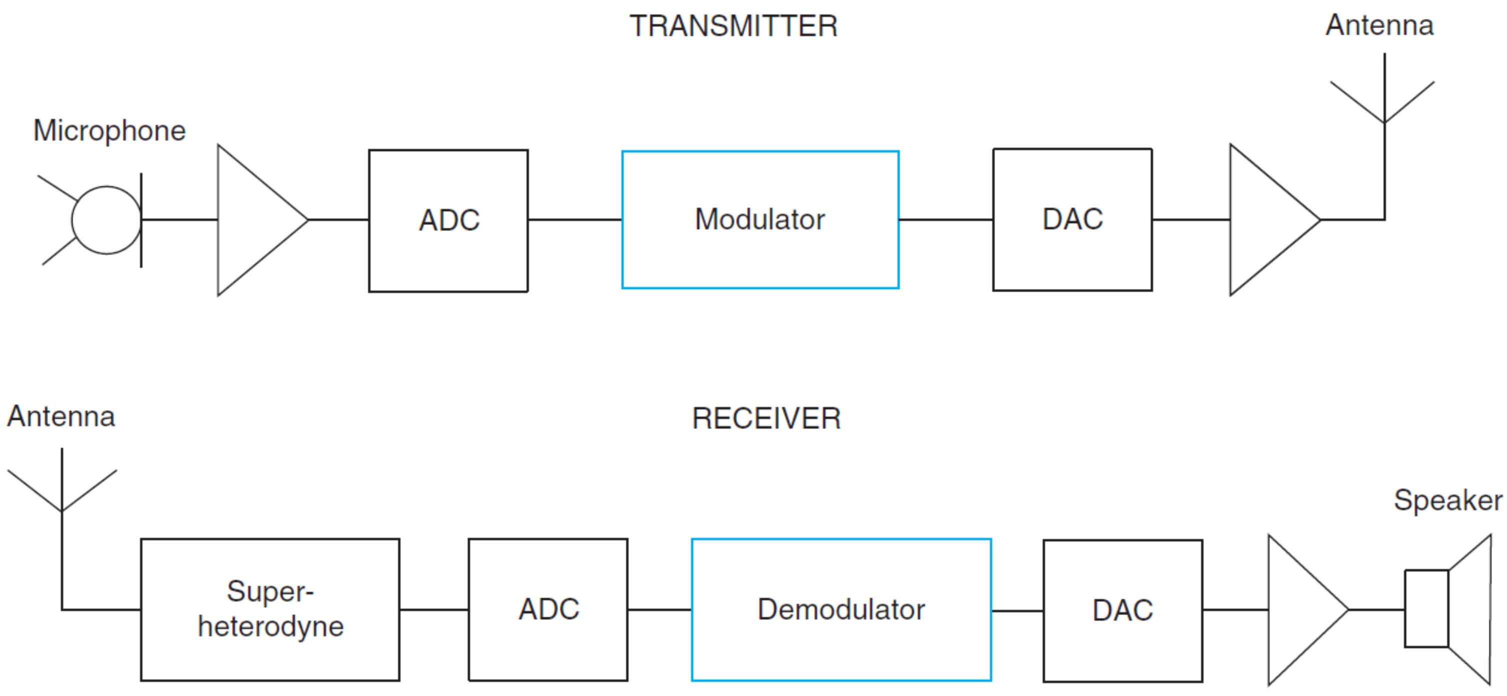
Examples of Signal Processing Applications (1)

- Compact-Disc Player
  - Analog sound signals
  - Sampled and stored in digital form
  - Read as digital and converted back to analog
  - High fidelity



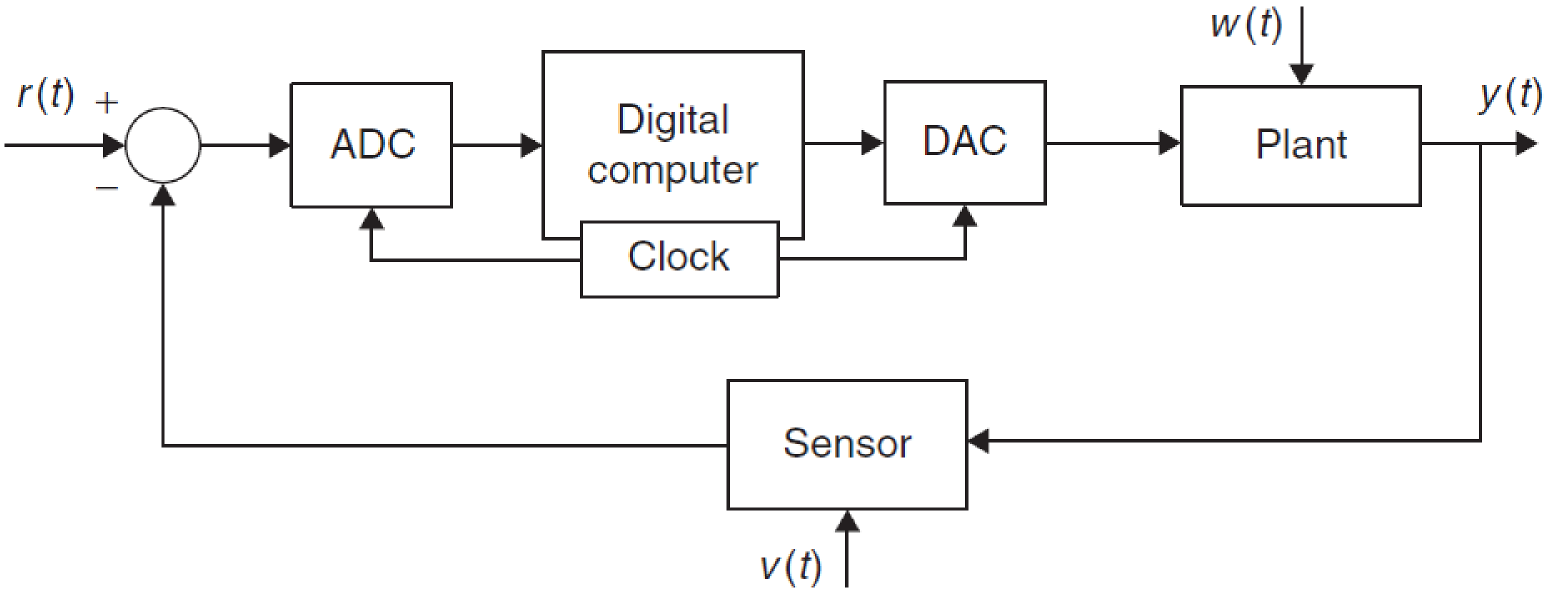
Examples of Signal Processing Applications (2)

- Software-Defined Radio and Cognitive Radio



Examples of Signal Processing Applications (3)

- Computer-Controlled Systems



Analog vs. Discrete

- Analog: Infinitesimal calculus (or just calculus)
  - Functions of continuous variables
  - Derivative
  - Integral
  - Differential equations
- Discrete: Finite calculus
  - Sequences
  - Difference
  - Summation
  - Difference equations

Real Life

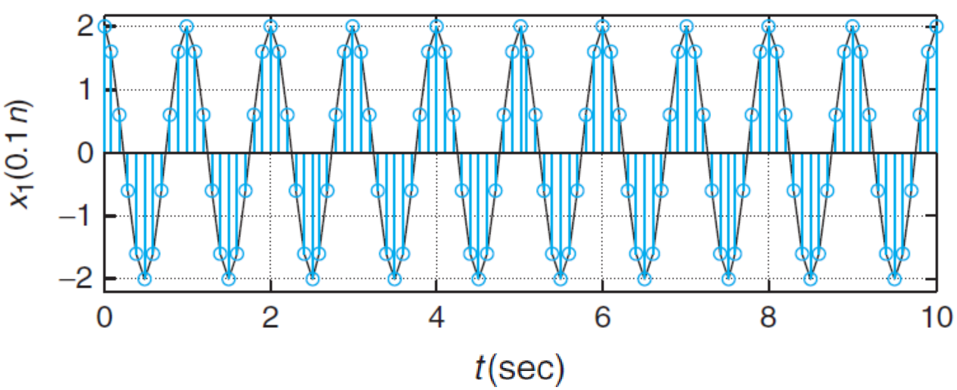
Computer

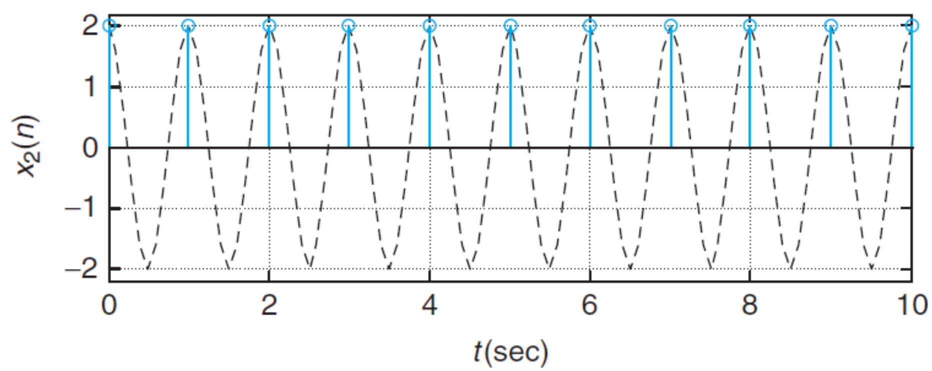
Continuous-Time and Discrete-Time Representations

- Discrete-time signal x[n] and corresponding analog signal x(t) are related by sampling:

$$x[n] = x(nT_s) = x(t)_{|t=nT_s}$$

- Ex: $x(t) = 2\cos(2\pi t), 0 \leq t \leq 10$





Derivatives and Finite Differences

- Derivative operator D

$$D[x(t)] = \frac{dx(t)}{dt} = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$

- Forward finite-difference operator Δ

$$\Delta[x(nT_s)] = x((n+1)T_s) - x(nT_s)$$

$$\Rightarrow$$

$$\Delta[x[n]] = x[n+1] - x[n]$$

- Operators on functions to give other functions
- Related by:

$$\frac{dx(t)}{dt}\Big|_{t=nT_s} = \lim_{T_s \to 0} \frac{\Delta[x(nT_s)]}{T_s}$$

Derivatives and Finite Differences: Example

- Consider the following 3 cases

$$x_0[n] = 2 \qquad \Delta[x_0[n]] = 2 - 2 = 0$$

$$x_1(t) = t \qquad \Delta[x_1[n]] = \Delta[n] = (n+1) - n = 1$$

$$x_2(t) = t^2 \qquad \Delta[x_2[n]] = \Delta[n^2] = (n+1)^2 - n^2 = 2n + 1$$

$$\text{derivative} \quad \Delta[x_2(0.01n)]/T_s = 10^{-2}(2n+1)$$

Whenever the rate of change of the signal is faster, difference gets closer to derivative by making T_s smaller

Integrals and Summations

- Integration D^{-1}

$$I(t) = \int_{t_0}^{t} x(\tau)d\tau$$

$$D[D^{-1}[x(t)]] = x(t).$$

$$\frac{dI(t)}{dt} = \lim_{h \to 0} \frac{I(t) - I(t-h)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{t-h}^{t} x(\tau)d\tau$$

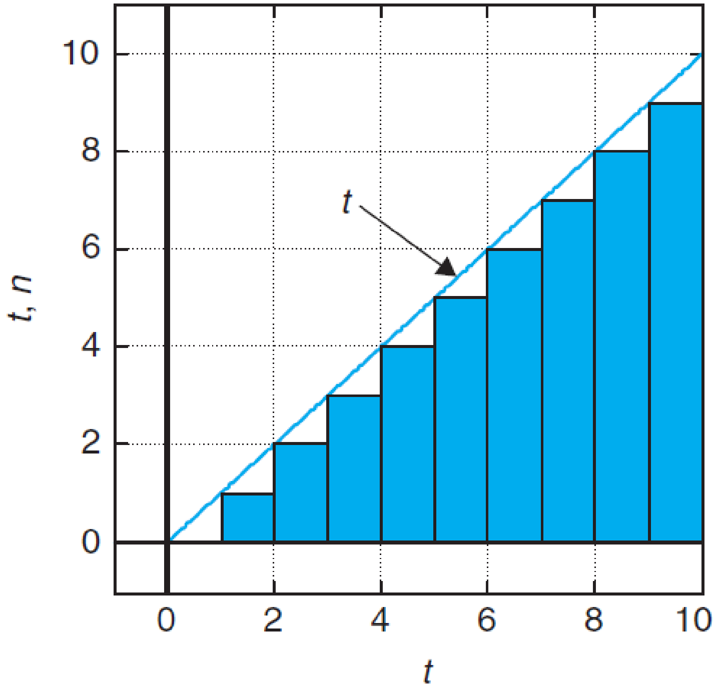
$$\approx \lim_{h \to 0} \frac{x(t) + x(t-h)}{2} = x(t)$$

- Computationally, integration is implemented by sums

$$\int_{0}^{10} t \ dt = \frac{t^2}{2} \Big|_{t=0}^{10} = 50.$$

Integrals and Summations: Example

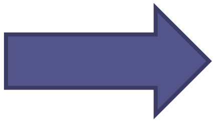
- Approximation of area under x(t)= t, 0<t<10
  - True result: $t^2/2 = 50$
  - Ts= 1: sum result= 45
    - Very poor approximation
  - Ts= 10^{-3}: sum result= 49.995
    - Much better approximation

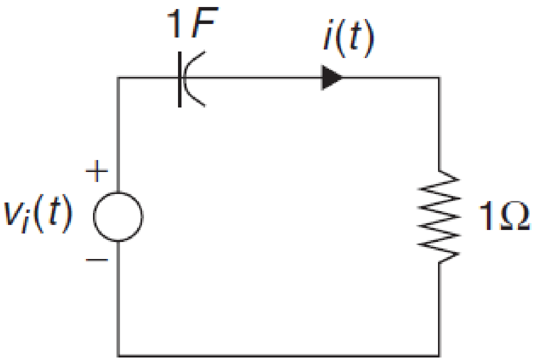


Differential and Difference Equations

- Differential equation characterizes the dynamics of a continuous-time system
  - approximated as linear constant-coefficient differential equations for simplification
- Solution: Analog computer

$$v_i(t) = v_c(t) + \frac{dv_c(t)}{dt}$$



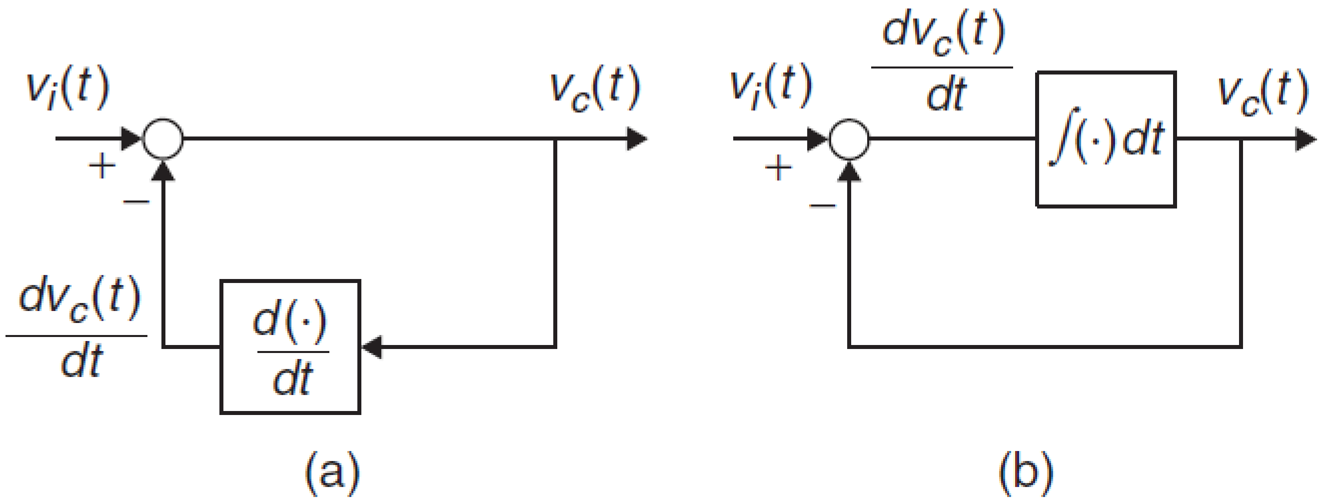


- Solution: Digital Computer
  - Convert to derivative to difference
  - Difference equation

Differential and Difference Equations: Block Diagram

- Realization of 1st order differential equation
  - Practical implementation using Op Amp circuits

$$v_i(t) = v_c(t) + \frac{dv_c(t)}{dt}$$



How to Obtain Difference Equations?

- Start from the differential equation:

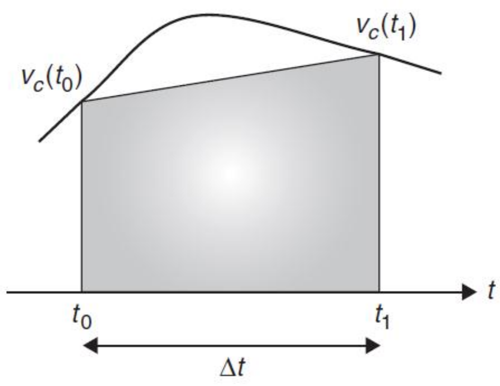
$$v_i(t) = v_c(t) + \frac{dv_c(t)}{dt}$$

$$v_c(t) = \int_0^t [v_i(\tau) - v_c(\tau)]d\tau + v_c(0) \qquad t \geq 0$$

$$v_c(t_1) - v_c(t_0) = \int_{t_0}^{t_1} v_i(\tau)d\tau - \int_{t_0}^{t_1} v_c(\tau)d\tau$$

$$v_c(t_1) - v_c(t_0) = [v_i(t_1) + v_i(t_0)]\frac{\Delta t}{2} - [v_c(t_1) + v_c(t_0)]\frac{\Delta t}{2}$$

$$\boxed{v_c(t_1)\left[1 + \frac{\Delta t}{2}\right] = [v_i(t_1) + v_i(t_0)]\frac{\Delta t}{2} + v_c(t_0)\left[1 - \frac{\Delta t}{2}\right]}$$



How to Obtain Difference Equations?

- Assuming $\Delta t = T$, let $t_1 = nT$, $t_0 = (n-1)T$, initial condition $v_c(0)=0$,

$$v_c(t_1)\left[1+\frac{\Delta t}{2}\right] = [v_i(t_1)+v_i(t_0)]\frac{\Delta t}{2} + v_c(t_0)\left[1-\frac{\Delta t}{2}\right]$$

$$\Downarrow$$

$$v_c(nT) = \frac{T}{2+T}[v_i(nT)+v_i((n-1)T)] + \frac{2-T}{2+T}v_c((n-1)T) \qquad n \geq 1$$

- First order linear difference equation with constant coefficients
    - Approximation of differential equation

Solution of Difference Equation

- Recursive solution
  - Obtain solution for n given solution for n-1
- Example: Solution for input $v_i(t) = 1 \text{ for } t \geq 0$

$$v_c(nT) = \begin{cases} 0 & n = 0 \\ \frac{2T}{2+T} + \frac{2-T}{2+T} v_c((n-1)T) & n \geq 1 \end{cases}$$

$$M = 2T/(2+T),\ K = (2-T)/(2+T)$$

$n = 0 \qquad v_c(0) = 0$

$n = 1 \qquad v_c(T) = M$

$n = 2 \qquad v_c(2T) = M + KM = M(1+K)$

$n = 3 \qquad v_c(3T) = M + K(M + KM) = M(1 + K + K^2)$

$n = 4 \qquad v_c(4T) = M + KM(1 + K + K^2) = M(1 + K + K^2 + K^3)$

$\ldots$

Complex Variables

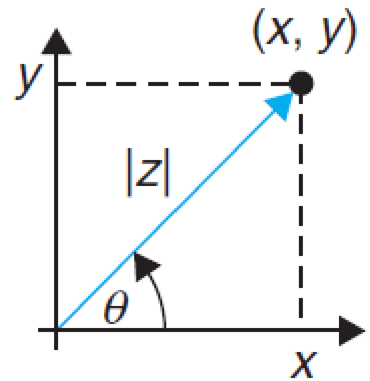
- Most of the theory of signals and systems is based on functions of a complex variable
- However, practical signals are functions of a real variable corresponding to time or space
- Complex variables represent mathematical tools that allow characteristics of signals to be defined in an easier to manipulate form
  - Example: phase of a sinusoidal signal

Complex Numbers and Vectors

- A complex number *z* represents any point (*x*, *y*):

$$z = x + j\,y,$$

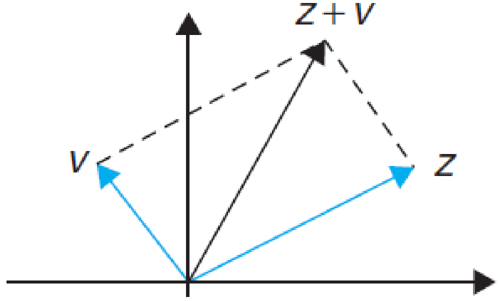
  - x =Re[z] (real part of z)
  - y =Im[z] (imaginary part of z)
  - j =Sqrt(-1)



- Vector representation
  - Rectangular or polar form $z = x + jy = |z|e^{j\theta}$
  - Magnitude $|\vec{z}| = \sqrt{x^2 + y^2} = |z|$ and Phase $\theta = \angle\vec{z} = \angle z$

Complex Numbers and Vectors

- Identical: use either depending on operation
  - Rectangular form for addition or subtraction
  - Polar form for multiplication or division
- Example: let $z = x + jy = |z|e^{j\angle z}$ and $v = p + jq = |v|e^{j\angle v}$



$$z + v = (x + p) + j(y + q)$$

$$zv = |z|e^{j\angle z}|v|e^{j\angle v} = |z||v|e^{j(\angle z + \angle v)}$$

$$zv = (x + jy)(p + jq) = (xp - yq) + j(xq + yp)$$

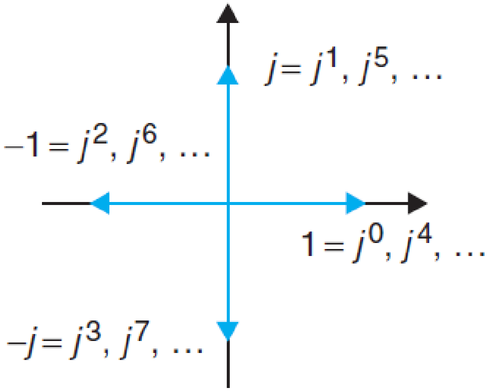
More Difficult

Complex Numbers and Vectors

- Powers of complex numbers: polar form

$$z^n = |z|^n e^{jn\angle z}$$

$$j^n = (-1)^{n/2} = \begin{cases} (-1)^m & n = 2m, \quad n \text{ even} \\ (-1)^m j & n = 2m+1, \quad n \text{ odd} \end{cases}$$



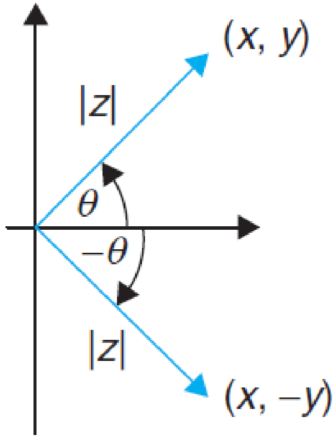
- Conjugate $z^* = x - jy = |z|e^{-j\angle z}$

(i) $z + z^* = 2x$ or $\mathcal{R}e[z] = 0.5[z + z^*]$

(ii) $z - z^* = 2jy$ or $\mathcal{I}m[z] = 0.5[z - z^*]$

(iii) $zz^* = |z|^2$ or $|z| = \sqrt{zz^*}$

(iv) $\dfrac{z}{z^*} = e^{j2\angle z}$ or $\angle z = -j0.5[\log(z) - \log(z^*)]$



Functions of a Complex Variable

- Just like real-valued functions
  - Example: Logarithm

$$v = \log(z) = \log(|z|e^{j\theta}) = \log(|z|) + j\theta$$

- Euler's identity

$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$

  - Proof: compute polar representation of R.H.S.

$$\cos(\theta) + j\sin(\theta) = 1e^{j\theta}$$

  - Example: $e^{\pm j\pi} = -1 \Rightarrow 1 + e^{j\pi} = 1 + e^{-j\pi} = 0$

Functions of a Complex Variable

- Starting from Euler's Identity, one can show:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

$$\Rightarrow$$

$$\cos(\theta) = \mathcal{R}e[e^{j\theta}] = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin(\theta) = \mathcal{I}m[e^{j\theta}] = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$e^{-\alpha} = \cosh(\alpha) - \sinh(\alpha)$$

$$\Rightarrow$$

$$\cos(j\alpha) = \frac{e^{-\alpha} + e^{\alpha}}{2} = \cosh(\alpha)$$

$$j\sin(j\alpha) = \frac{e^{-\alpha} - e^{\alpha}}{2} = -\sinh(\alpha)$$

$$\cos^2(\theta) = \left[\frac{e^{j\theta} + e^{-j\theta}}{2}\right]^2 = \frac{1}{4}[2 + e^{j2\theta} + e^{-j2\theta}] = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

Phasors and Sinusoidal Steady State

- A sinusoid is a periodic signal represented by,

$$v(t) = A\cos(\Omega_0 t + \psi)$$

- If one knows the frequency, cosine is characterized by its amplitude and phase.
- Define ***Phasor*** as complex number characterized by amplitude and the phase of a cosine signal

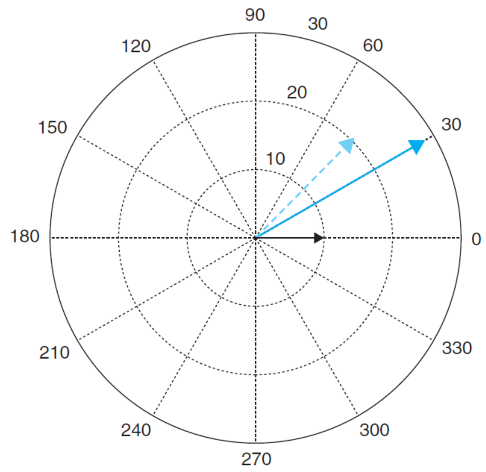
$$V = Ae^{j\psi} = A\cos(\psi) + jA\sin(\psi) = A\angle\psi$$

  - Such that

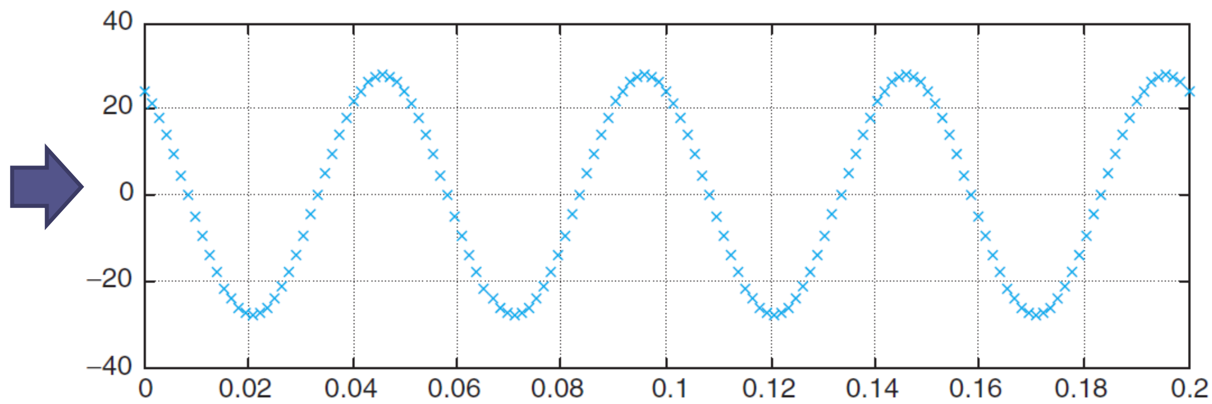
$$v(t) = \mathcal{R}e[Ve^{j\Omega_0 t}] = \mathcal{R}e[Ae^{j(\Omega_0 t + \psi)}] = A\cos(\Omega_0 t + \psi)$$

Phasor Connection

- Fundamental property of a circuit made up of constant resistors, capacitors, and inductors is that its response to a sinusoid is also a sinusoid of same frequency in steady state
  - Circuit of linear and time-invariant nature

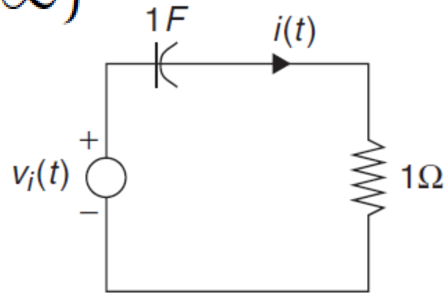


Phasor 1 + Phasor 2



Phasor Connection

- Example: Steady state solution of RC circuit with input $v_i(t) = A\cos(\Omega_0 t)$
  - Assume that the steady-state response of this circuit is also a sinusoid (i.e., $v_c(t)$ as $t \to \infty$)
  - Then, we can let $v_c(t) = C\cos(\Omega_0 t + \psi)$
  - Substitute in $v_i(t) = \dfrac{dv_c(t)}{dt} + v_c(t)$



$$A\cos(\Omega_0 t) = -C\Omega_0 \sin(\Omega_0 t + \psi) + C\cos(\Omega_0 t + \psi)$$

$$= C\Omega_0 \cos(\Omega_0 t + \psi + \pi/2) + C\cos(\Omega_0 t + \psi)$$

$$= C\sqrt{1 + \Omega_0^2}\cos(\Omega_0 t + \psi + \tan^{-1}(C\Omega_0/C))$$

$$C = \frac{A}{\sqrt{1 + \Omega_0^2}}$$

$$\psi = -\tan^{-1}(\Omega_0)$$

Phasor Connection

- Same solution using phasor notation:

$$V_c = Ce^{j\psi} \qquad v_c(t) = \mathcal{R}e\left[V_c e^{j\Omega_0 t}\right]$$

$$V_i = Ae^{j0} \qquad v_i(t) = \mathcal{R}e\left[V_i e^{j\Omega_0 t}\right]$$

$$\frac{dv_c(t)}{dt} = \frac{d\mathcal{R}e[V_c e^{j\Omega_0 t}]}{dt} = \mathcal{R}e\left[V_c \frac{de^{j\Omega_0 t}}{dt}\right] = \mathcal{R}e\left[j\Omega_0 V_c e^{j\Omega_0 t}\right]$$

  - Differential equation: $\mathcal{R}e\left[V_c(1 + j\Omega_0)e^{j\Omega_0 t}\right] = \mathcal{R}e\left[Ae^{j\Omega_0 t}\right]$
  - By comparison, get V_c and hence C and ψ

Problem Assignments

- Problems: 0.1, 0.2, 0.3, 0.11, 0.12, 0.15, 0.23
- Partial Solutions available from the student section of the textbook web site