Signals and Systems – Chapter 2

Continuous-Time Systems

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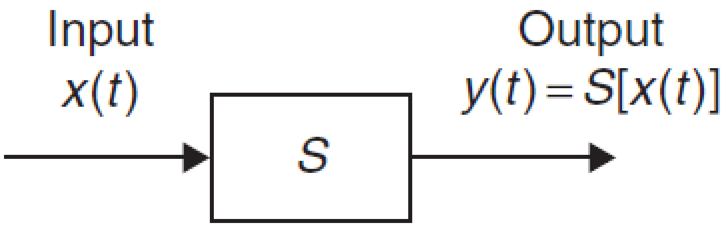
Overview of Chapter 2

- Systems and their classification
- Linear time-invariant systems

“System” Concept

- Mathematical transformation of an input signal (or signals) into an output signal (or signals)
  - Idealized model of the physical device or process



- Examples:
  - Electrical/electronic circuits
- In practice, the model and the mathematical representation are not unique

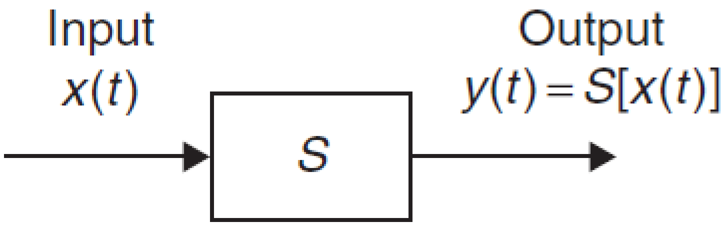
System Classification

- Static or dynamic systems
  - Capability of storing energy, or remembering state
- Lumped- or distributed-parameter systems
- Passive or active systems
  - Ex: circuits elements
- Continuous time, discrete time, digital, or hybrid systems
  - According to type of input/output signals

LTI Continuous-Time Systems

- A continuous-time system is a system in which the signals at its input and output are continuous-time signals

$x(t)$	$\Rightarrow$	$y(t) = \mathcal{S}[x(t)]$
Input		Output

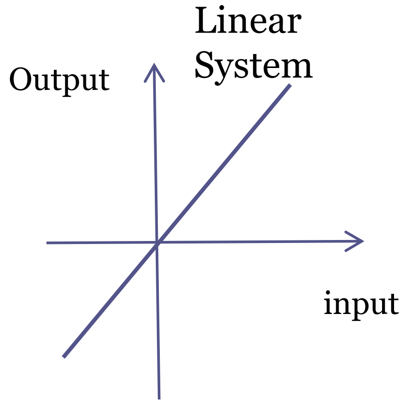


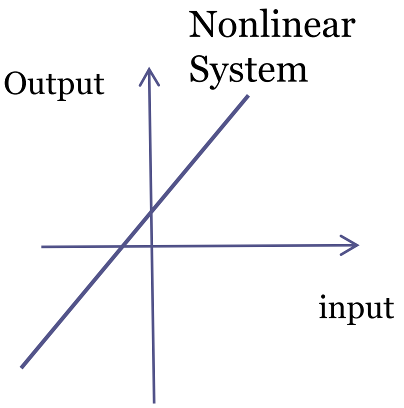
Linearity

- A linear system is a system in which the *superposition* holds
  - Scaling
  - Additivity

$$S[\alpha x(t) + \beta v(t)] = S[\alpha x(t)] + S[\beta v(t)]$$
$$= \alpha S[x(t)] + \beta S[v(t)]$$

- Examples:
  - $y(x)$= a x ➡ Linear
  - $y(x)$= a x + b ➡ Nonlinear





Linearity - Examples

- Show that the following systems are nonlinear:

$$(i)\ y(t) = |x(t)|$$

$$(ii)\ z(t) = \cos(x(t)) \ \text{ assuming } \ |x(t)| \leq 1$$

$$(iii)\ v(t) = x^2(t)$$

  - where $x(t)$ is the input and $y(t)$, $z(t)$, and $v(t)$ are the outputs.

Whenever the explicit relation between the input and the output of a system is represented by a nonlinear expression the system is nonlinear

Linearity - Examples

- Consider each of the components of an RLC circuit and determine under what conditions they are linear.
  - R

$$v(t) = Ri(t)$$

  - C

$$i(t) = C dv_c(t)/dt \qquad v_c(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + v_c(0)$$

  - L

$$v(t) = \frac{d\phi(t)}{dt} = L\frac{di_L(t)}{dt} \qquad i_L(t) = \frac{1}{L}\int_0^t v(\tau)d\tau + i_L(0)$$

Linearity - Examples

- Op Amp
  - Linear or nonlinear region

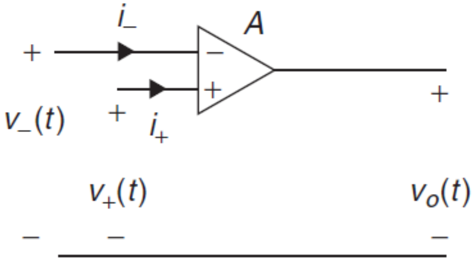
$$v_o(t) = Av_d(t) \qquad -\Delta V \le v_d(t) \le \Delta V$$

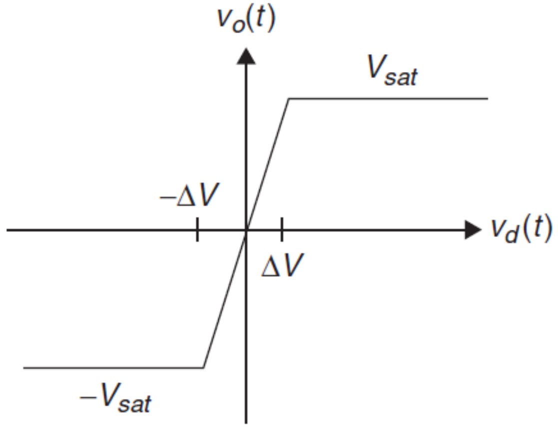
- Virtual short

$$A \to \infty \quad R_{in} \to \infty$$

$$i_- = i_+ = 0$$

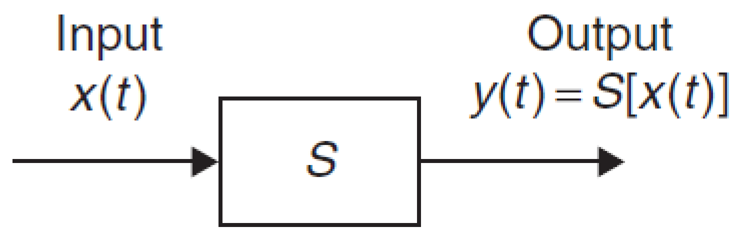
$$v_d(t) = v_+(t) - v_-(t) = 0$$





Time Invariance

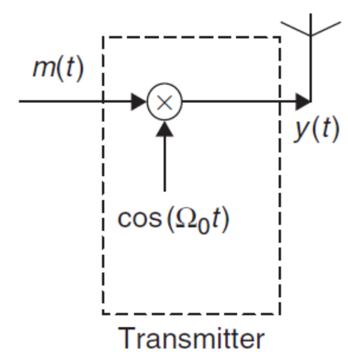
- System S does not change with time
  - System does not age—its parameters are constant



$$x(t) \quad \Rightarrow \quad y(t) = \mathcal{S}[x(t)]$$

$$x(t \mp \tau) \quad \Rightarrow \quad y(t \mp \tau) = \mathcal{S}[x(t \pm \tau)]$$

- Example: AM modulation

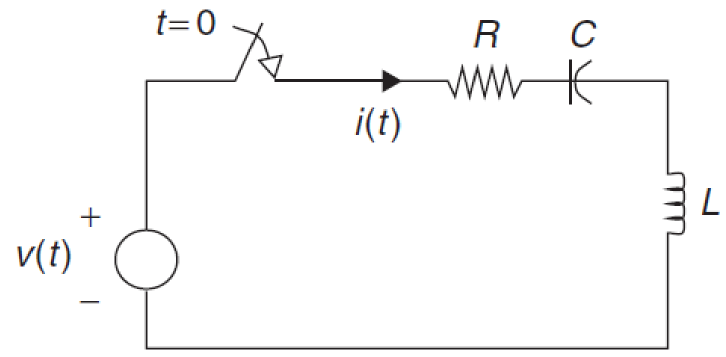


$$y(t) = \cos(\Omega_0 t)x(t)$$

RLC Circuits

- Kirchhoff's voltage law,

$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(\tau)d\tau$$



d/dt

$$\frac{dv(t)}{dt} = R\frac{di(t)}{dt} + L\frac{d^2i(t)}{dt^2} + \frac{1}{C}i(t)$$

- Second-order differential equation with constant coefficients
  - Input the voltage source *v(t)*
  - Output the current *i(t)*

Representation of Systems by Differential Equations

- Given a dynamic system represented by a linear differential equation with constant coefficients:

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} + \cdots + \frac{d^N y(t)}{dt^N} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_M \frac{d^M x(t)}{dt^M} \qquad t \geq 0$$

  - N initial conditions: $y(0)$, $d^k y(t)/dt^k|_{t=0}$ for $k = 1, \ldots, N-1$
  - Input $x(t)=0$ for $t < 0$,
- Complete response $y(t)$ for $t>=0$ has two parts:
  - Zero-state response
  - Zero-input response

$$y(t) = y_{zs}(t) + y_{zi}(t)$$

Representation of Systems by Differential Equations

- Linear Time-Invariant Systems
  - System represented by linear differential equation with constant coefficients
  - Initial conditions are all zero
  - Output depends exclusively on input only

$$y(t) = y_{zs}(t)$$

- Nonlinear Systems
  - Nonzero initial conditions means nonlinearity
  - Can also be time-varying

$$y(t) = y_{zs}(t) + y_{zi}(t)$$

Representation of Systems by Differential Equations

- Define derivative operator D as,

$$D^n[y(t)] = \frac{d^n y(t)}{dt^n} \qquad n > 0, \text{ integer}$$

$$D^0[y(t)] = y(t)$$

- Then,

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} + \cdots + \frac{d^N y(t)}{dt^N} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_M \frac{d^M x(t)}{dt^M} \qquad t \geq 0$$

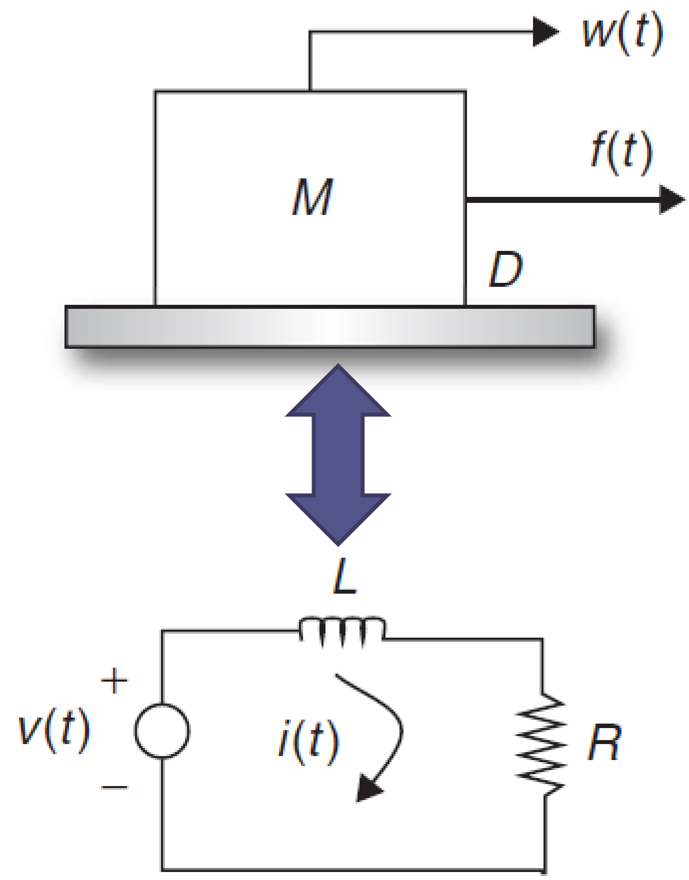
$$\Downarrow$$

$$(a_0 + a_1 D + \cdots + D^N)[y(t)] = (b_0 + b_1 D + \cdots + b_M D^M)[x(t)] \qquad t \geq 0$$

Analog Mechanical Systems

Table 2.1 Equivalences in Mechanical and Electrical Systems

Mechanical System	Electrical System
force f(t)	voltage v(t)
velocity w(t)	current i(t)
mass M	inductance L
damping D	resistance R
compliance K	capacitance C



Application of Superposition and Time Invariance

- The computation of the output of an LTI system is simplified when the input can be represented as the combination of signals for which we know their response.
  - Using superposition and time invariance properties

If $\mathcal{S}$ is the transformation corresponding to an LTI system, so that the response of the system is

$$y(t) = \mathcal{S}[x(t)] \text{ for an input } x(t)$$

then we have that

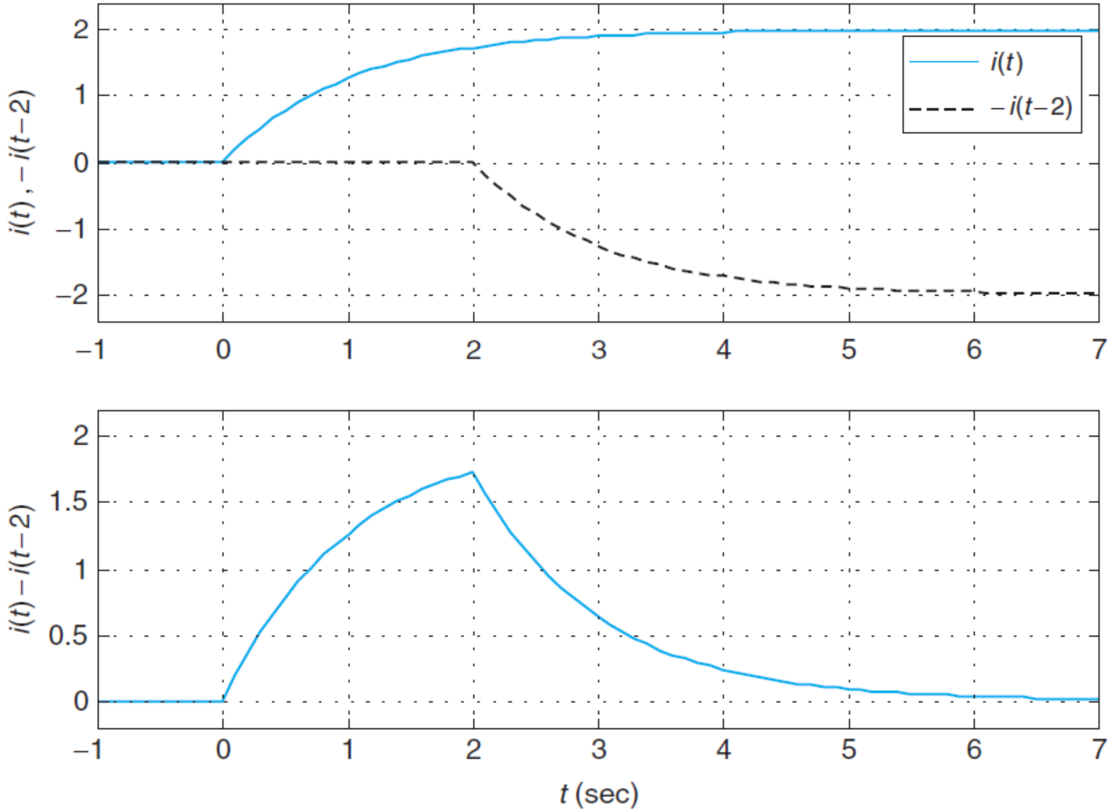
$$\mathcal{S}\left[\sum_k A_k x(t-\tau_k)\right] = \sum_k A_k \mathcal{S}[x(t-\tau_k)] = \sum_k A_k y(t-\tau_k)$$

$$\mathcal{S}\left[\int g(\tau)x(t-\tau)d\tau\right] = \int g(\tau)\mathcal{S}[x(t-\tau)]d\tau = \int g(\tau)y(t-\tau)d\tau$$

In the next section we will see that this property allows us to find the response of a linear time-invariant system due to any signal, if we know the response of the system to an impulse signal.

Application of Superposition and Time Invariance: Example

- Example 1: Given the response of an RL circuit to a unit-step source $u(t)$, find the response to a pulse $v(t) = u(t) - u(t-2)$



Convolution Integral

- Generic representation of a signal:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

- The impulse response of an analog LTI system, $h(t)$, is the output of the system corresponding to an impulse $\delta(t)$ as input, and zero initial conditions
- The response of an LTI system S represented by its impulse response $h(t)$ to any signal $x(t)$ is given by:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$
$$= [x * h](t) = [h * x](t)$$

Convolution Integral

Convolution Integral: Observations

- Impulse response is fundamental in the characterization of linear time-invariant systems
- Any system characterized by the convolution integral is linear and time invariant by the above construction
- The convolution integral is a general representation of LTI systems, given that it was obtained from a generic representation of the input signal
- Given that a system represented by a linear differential equation with constant coefficients and no initial conditions, or input, before t=0 is LTI, one should be able to represent that system by a convolution integral after finding its impulse response $h(t)$

Convolution Integral: Example

- Example: Obtain the impulse response of a capacitor and use it to find its unit-step response by means of the convolution integral. Let $C = 1$ F.

$$v_c(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$$

letting the input $i(t) = \delta(t)$

$$h(t) = \frac{1}{C}\int_0^t \delta(\tau)d\tau = \frac{1}{C} \qquad t > 0$$

$$v_c(t) = \int_{-\infty}^{\infty} h(t-\tau)i(\tau)d\tau = \int_{-\infty}^{\infty} \frac{1}{C}u(t-\tau)u(\tau)d\tau$$

Causality

- A continuous-time system S is called causal if:
  - Whenever the input $x(t)=0$ and there are no initial conditions, the output is $y(t)=0$
  - The output $y(t)$ does not depend on future inputs
- For a value $\tau > 0$, when considering causality it is helpful to think of:
  - Time t (the time at which the output $y(t)$ is being computed) as the *present*
  - Times $t-\tau$ as the *past*
  - Times $t+\tau$ as the *future*

Causality

An LTI system represented by its impulse response $h(t)$ is *causal* if

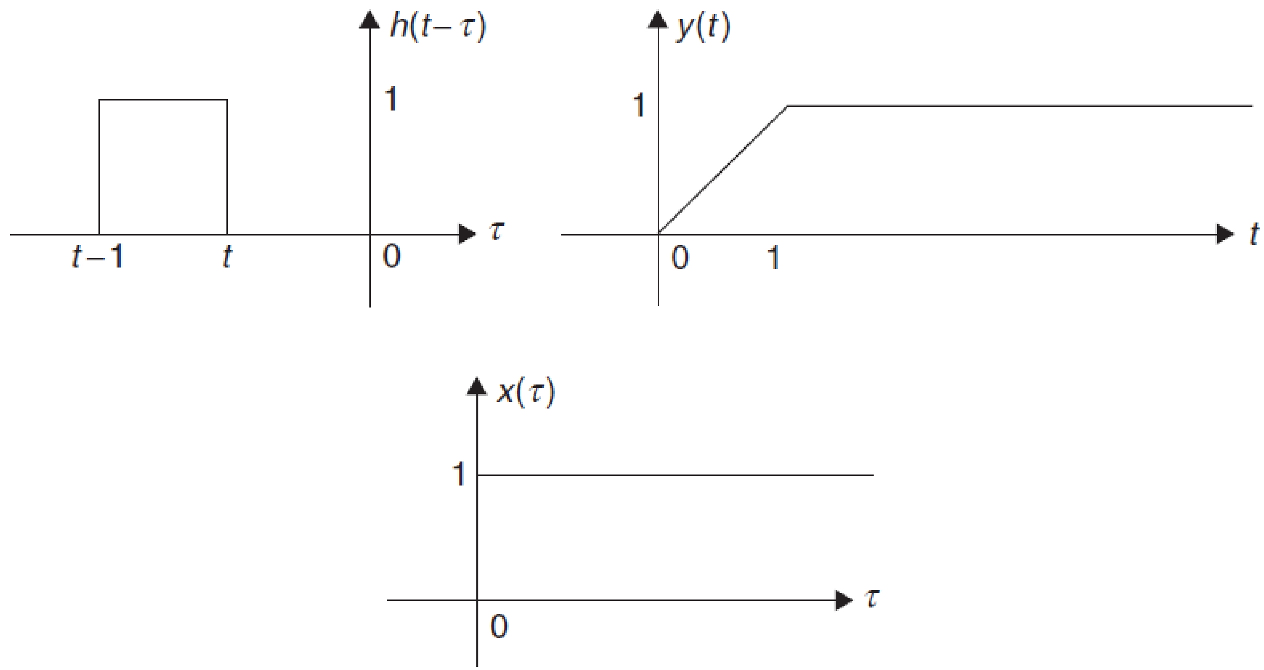
$$h(t) = 0 \qquad \text{for } t < 0$$

The output of a causal LTI system with a causal input $x(t)$ (i.e., $x(t) = 0$ for $t < 0$) is

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau$$

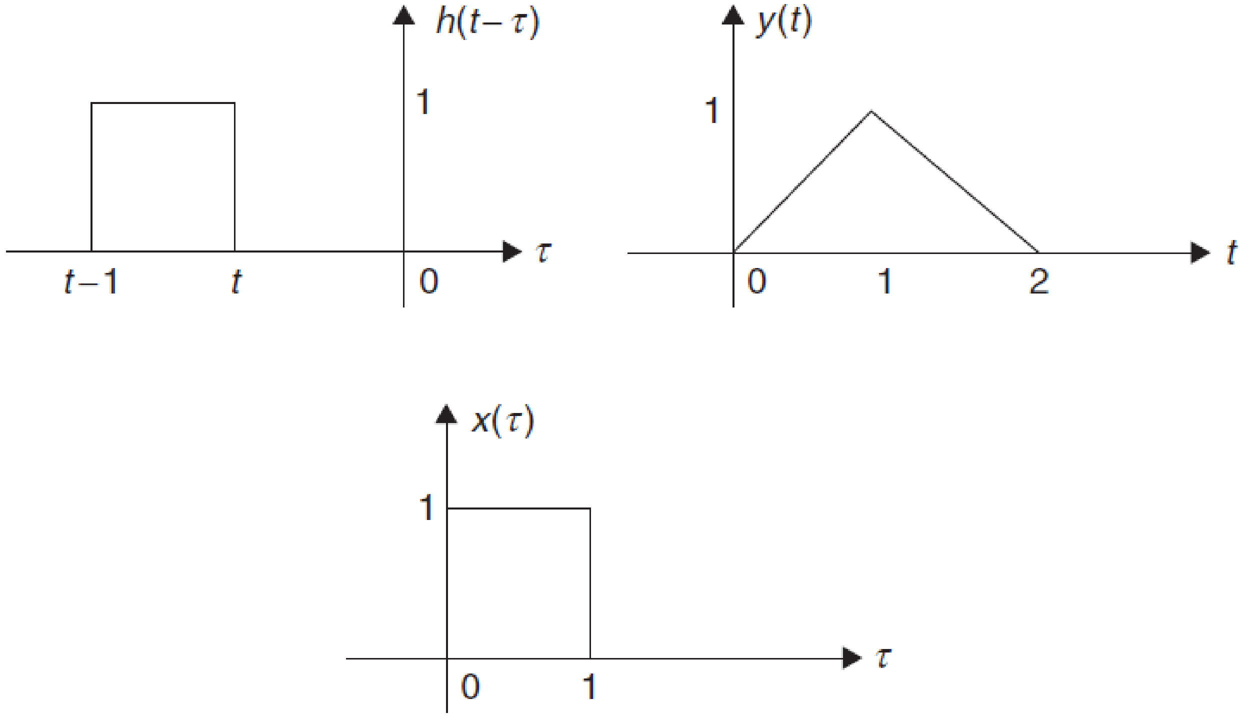
Graphical Computation of Convolution Integral

- Example 1: Graphically find the unit-step $y(t)$ response of an averager, with T=1 sec, which has an impulse response $h(t)= u(t)-u(t-1)$



Graphical Computation of Convolution Integral

- Example 2: Consider the graphical computation of the convolution integral of two pulses of the same duration

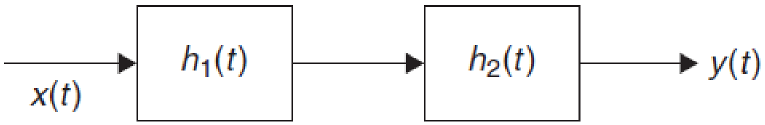


The length of the support of $y(t) = [x * h](t)$ is equal to the sum of the lengths of the supports of $x(t)$ and $h(t)$.

Interconnection of Systems—Block Diagrams

- (a) Cascade (commutative)

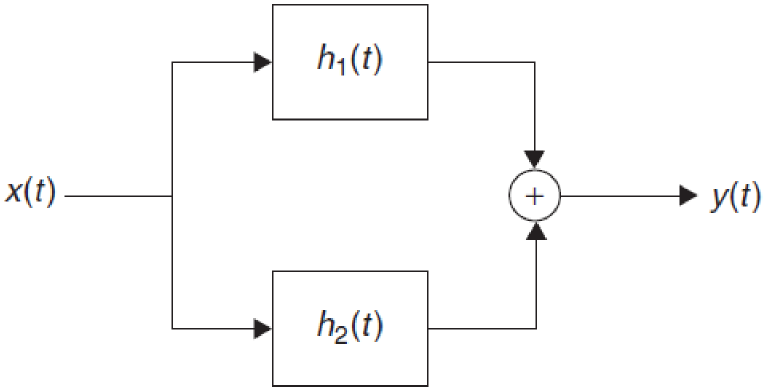
$$h(t) = [h_1 * h_2](t) = [h_2 * h_1](t)$$



(a)

- (b) Parallel (distributive)

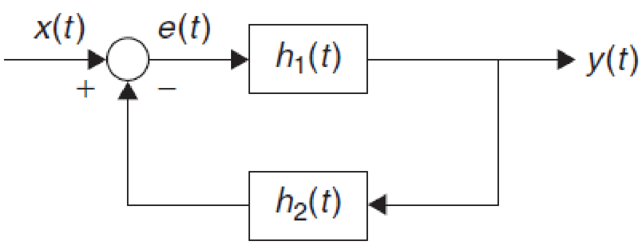
$$h(t) = h_1(t) + h_2(t)$$



(b)

- (c) Feedback

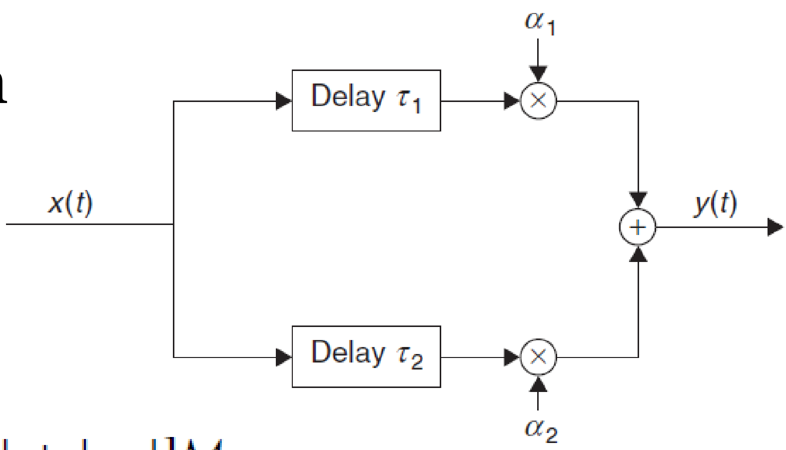
$$h(t) = [h_1 - h * h_1 * h_2](t)$$



(c)

Bounded-Input Bounded-Output Stability (BIBO)

- For a bounded (i.e., well-behaved) input $x(t)$, the output of a BIBO stable system $y(t)$ is also bounded
- An LTI system with an absolutely integrable impulse response is BIBO stable
- Example: Multi-echo path system



$$|y(t)| \leq |\alpha_1||x(t-\tau_1)| + |\alpha_2||x(t-\tau_2)| < [|\alpha_1| + |\alpha_2|]M$$

Problem Assignments

- Problems: 2.3, 2.4, 2.8, 2.9, 2.10, 2.12, 2.14
- Partial Solutions available from the student section of the textbook web site