Signals and Systems – Chapter 3

The Laplace Transform -Part 1

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Eigenfunctions of LTI Systems

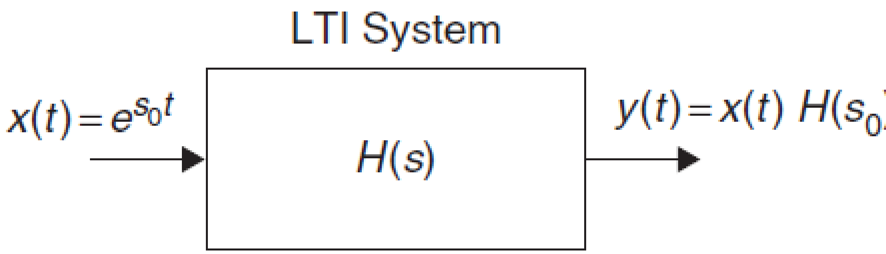
- Consider as the input of an LTI system the complex signal $x(t) = e^{s_0 t}$ $\quad s_0 = \sigma_0 + j\Omega_0$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s_0(t-\tau)}d\tau$$

$$= e^{s_0 t}\int_{-\infty}^{\infty} h(\tau)e^{-\tau s_0}d\tau = x(t)H(s_0)$$

$$\boxed{H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-\tau s}d\tau}$$

Laplace Transform of h(t)!



Eigenfunctions of LTI Systems

An input $x(t) = e^{s_0 t}$, $s_0 = \sigma_0 + j\Omega_0$, is called an eigenfunction of an LTI system with impulse response $h(t)$ if the corresponding output of the system is

$$y(t) = x(t) \int_{-\infty}^{\infty} h(t)e^{-s_0 t} = x(t)H(s_0)$$

where $H(s_0)$ is the Laplace transform of $h(t)$ computed at $s = s_0$. This property is only valid for LTI systems—it is not satisfied by time-varying or nonlinear systems.

- Suppose a signal x(t) is expressed as a sum of complex exponentials in s

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st} ds \Rightarrow y(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\left[H(s)e^{st}\right] ds = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} Y(s)e^{st} ds$$

$$\boxed{y(t) = [x * h](t) \quad \Leftrightarrow \quad Y(s) = X(s)H(s)}$$

The Two-Sided Laplace Transform

The two-sided Laplace transform of a continuous-time function $f(t)$ is

$$F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-st}dt \qquad s \in \text{ROC} \tag{3.2}$$

where the variable $s = \sigma + j\Omega$, with Ω as the frequency in rad/sec and σ as a damping factor. ROC stands for the region of convergence—that is, where the integral exists.

The inverse Laplace transform is given by

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds \qquad \sigma \in \text{ROC} \tag{3.3}$$

The Two-Sided Laplace Transform

- Laplace transform F(s) provides a representation of f(t) in the s-domain, which in turn can be inverted back into the original time-domain function (1:1)
- Laplace transform of impulse response of an LTI system h(t) is H(s) and is called the system or *transfer function*
- Region of Convergence: region in s where transform exists

For the Laplace transform of $f(t)$ to exist we need that

$$\left|\int_{-\infty}^{\infty} f(t)e^{-st}dt\right| = \left|\int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\Omega t}dt\right|$$

$$\leq \int_{-\infty}^{\infty} |f(t)e^{-\sigma t}|\, dt < \infty$$

or that $f(t)e^{-\sigma t}$ be absolutely integrable. This may be possible by choosing an appropriate σ even in the case when $f(t)$ is not absolutely integrable. The value chosen for σ determines the ROC of $F(s)$; the frequency Ω does not affect the ROC.

The Two-Sided Laplace Transform

- ROC

The Laplace transform of a

- Finite support function (i.e., $f(t) = 0$ for $t < t_1$ and $t > t_2$, for $t_1 < t_2$) is

$$\mathcal{L}[f(t)] = \mathcal{L}\left[f(t)[u(t - t_1) - u(t - t_2)]\right] \qquad \text{whole } s\text{-plane}$$

- Causal function (i.e., $f(t) = 0$ for $t < 0$) is

$$\mathcal{L}[f(t)u(t)] \qquad \mathcal{R}_c = \{(\sigma, \Omega) : \sigma > \max\{\sigma_i\}, -\infty < \Omega < \infty\}$$

- Anti-causal function (i.e., $f(t) = 0$ for $t > 0$) is

$$\mathcal{L}[f(t)u(-t)] \qquad \mathcal{R}_{ac} = \{(\sigma, \Omega) : \sigma < \min\{\sigma_i\}, -\infty < \Omega < \infty\}$$

- Noncausal function (i.e., $f(t) = f_{ac}(t) + f_c(t) = f(t)u(-t) + f(t)u(t)$) is

$$\mathcal{L}[f(t)] = \mathcal{L}[f_{ac}(-t)u(t)]_{(-s)} + \mathcal{L}[f_c(t)u(t)] \qquad \mathcal{R}_c \bigcap \mathcal{R}_{ac}$$

The One-Sided Laplace Transform

The *one-sided Laplace transform* is defined as

$$F(s) = \mathcal{L}[f(t)u(t)] = \int_{0-}^{\infty} f(t)u(t)e^{-st}\,dt \qquad (3.8)$$

where $f(t)$ is either a causal function or made into a causal function by the multiplication by $u(t)$. The one-sided Laplace transform is of significance given that most of the applications deal with causal systems and signals, and that any signal or system can be decomposed into causal and anti-causal components requiring only the computation of one-sided Laplace transforms.

Linearity

For signals $f(t)$ and $g(t)$, with Laplace transforms $F(s)$ and $G(s)$, and constants a and b, we have the Laplace transform is linear:

$$\mathcal{L}[af(t)u(t) + bg(t)u(t)] = aF(s) + bG(s)$$

$$\mathcal{L}[af(t)u(t) + bg(t)u(t)] = \int_0^\infty [af(t) + bg(t)]u(t)e^{-st}dt$$

$$= a\int_0^\infty f(t)u(t)e^{-st}dt + b\int_0^\infty g(t)u(t)e^{-st}dt$$

$$= a\mathcal{L}[f(t)u(t)] + b\mathcal{L}[g(t)(t)]$$

Differentiation

For a signal $f(t)$ with Laplace transform $F(s)$ its one-sided Laplace transform of its first-and second-order derivatives are

$$\mathcal{L}\left[\frac{df(t)}{dt}u(t)\right] = sF(s) - f(0-) \qquad (3.11)$$

$$\mathcal{L}\left[\frac{df(t)}{dt}u(t)\right] = \int_{0-}^{\infty} \frac{df(t)}{dt} e^{-st} dt$$

integral is evaluated by parts

$$\int w dv = wv - \int v dw$$

$$\int_{0-}^{\infty} \frac{df(t)}{dt} e^{-st} dt = e^{-st} f(t) \Big|_{0-}^{\infty} - \int_{0-}^{\infty} f(t)(-se^{-st}) dt$$

$$= s \int_{0-}^{\infty} f(t) e^{-st} dt - f(0-)$$

$$= sF(s) - f(0-)$$

Integration

The Laplace transform of the integral of a causal signal $y(t)$ is given by

$$\mathcal{L}\left[\int_0^t y(\tau)d\tau \;\; u(t)\right] = \frac{Y(s)}{s} \tag{3.14}$$

$$f(t) = \int_0^t y(\tau)d\tau u(t) \quad \Rightarrow \quad \frac{df(t)}{dt} = y(t)u(t)$$

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0) = Y(s) \quad \Rightarrow \quad F(s) = \mathcal{L}\left[\int_0^t y(\tau)d\tau\right] = \frac{Y(s)}{s}$$

Time Shifting

If the Laplace transform of $f(t)u(t)$ is $F(s)$, the Laplace transform of the time-shifted signal $f(t-\tau)u(t-\tau)$ is

$$\mathcal{L}[f(t-\tau)u(t-\tau)] = e^{-\tau s}F(s) \tag{3.15}$$

- Simply shown by a change of variables

Convolution Integral

The Laplace transform of the convolution integral of a causal signal $x(t)$, with Laplace transforms $X(s)$, and a causal impulse response $h(t)$, with Laplace transform $H(s)$, is given by

$$\mathcal{L}[(x * h)(t)] = X(s)H(s) \qquad (3.16)$$

$$y(t) = \int_0^\infty x(\tau)h(t-\tau)d\tau \qquad t \geq 0$$

$$Y(s) = \mathcal{L}\left[\int_0^\infty x(\tau)h(t-\tau)d\tau\right] = \int_0^\infty \left[\int_0^\infty x(\tau)h(t-\tau)d\tau\right] e^{-st}dt$$

$$= \int_0^\infty x(\tau)\left[\int_0^\infty h(t-\tau)\; e^{-s(t-\tau)}\; dt\right] e^{-s\tau}d\tau = X(s)H(s)$$

$$\boxed{H(s) = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[\,x(t)]} = \frac{\mathcal{L}[\text{ output }]}{\mathcal{L}[\text{ input }]}}$$

Laplace Transformation Table

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s, ROC
1.	$\delta(t)$	1, whole s-plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
4.	$e^{-at}u(t),\ a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
7.	$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A\, e^{-at}\cos(\Omega_0 t + \theta)u(t),\ a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!}\, t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!}\, t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!}\, t^{N-1}e^{-at}\cos(\Omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

Laplace Transform Properties

Table 3.2 Basic Properties of One-Sided Laplace Transforms

Causal functions and constants	$\alpha f(t),\ \beta g(t)$	$\alpha F(s),\ \beta G(s)$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Time shifting	$f(t - \alpha)$	$e^{-\alpha s} F(s)$
Frequency shifting	$e^{\alpha t} f(t)$	$F(s - \alpha)$
Multiplication by t	$t\, f(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
Second derivative	$\frac{d^2 f(t)}{dt^2}$	$s^2 F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t)\ \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0+) = \lim_{s\to\infty} sF(s)$	
Final value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$	

Example 1

Find the Laplace transforms of $\delta(t)$, $u(t)$, and a pulse $p(t) = u(t) - u(t-1)$.

$$\mathcal{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t)e^{-s0}dt = \int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$U(s) = \mathcal{L}[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt = \int_{0}^{\infty} e^{-\sigma t}e^{-j\Omega t}dt \Rightarrow U(s) = \frac{e^{-st}}{-s}\Big|_{t=0}^{\infty} = \frac{1}{s}$$

$$P(s) = \mathcal{L}[u(t+1) - u(t-1)] = \int_{-1}^{1} e^{-st}dt = \frac{-e^{-st}}{s}\Big|_{t=-1}^{1} = \frac{1}{s}[e^{s} - e^{-s}] = \frac{e^{s}}{s}[1 - e^{-2s}]$$

Example 2

- Compute H(s) for:

$$h(t) = e^{-t}u(t) + e^{2t}u(-t)$$
$$= h_c(t) + h_{ac}(t)$$

- Using table:
  - causal component: $$H_c(s) = \frac{1}{s+1} \qquad \sigma > -1$$
  - Anti-causal component: $$\mathcal{L}[h_{ac}(t)] = \mathcal{L}[h_{ac}(-t)u(t)]_{(-s)} = \frac{1}{-s+2}$$ $$\sigma < 2$$

$$\Downarrow$$

$$H(s) = \frac{1}{s+1} + \frac{1}{-s+2} = \frac{-3}{(s+1)(s-2)} \qquad -1 < \sigma < 2$$

Example 3

Compute the Laplace transform of the ramp function $r(t) = tu(t)$ and use it to find the Laplace of a triangular pulse $\Lambda(t) = r(t+1) - 2r(t) + r(t-1)$.

$$R(s) = \int_0^\infty te^{-st}dt = \frac{e^{-st}}{s^2}(-st-1)\,\Big|_{t=0}^{\infty} = \frac{1}{s^2} \qquad \sigma > 0$$

$$\Rightarrow \qquad \Lambda(s) = \frac{1}{s^2}[e^s - 2 + e^{-s}] \qquad \sigma > 0$$

Example 4

- Use the differentiation property to compute the Laplace transformation of $\delta(t)$, $u(t)$, and $r(t)$ starting from R(s) derived in Example 3

$$\mathcal{L}[r(t)] = \frac{1}{s^2}$$

$$\mathcal{L}\left[u(t) = \frac{dr(t)}{dt}\right] = s\frac{1}{s^2} = \frac{1}{s}$$

$$\mathcal{L}\left[\delta(t) = \frac{du(t)}{dt}\right] = s\frac{1}{s} = 1$$

Example 5

- Let y(t) be a causal signal. Compute Y(s) given that

$$\int_0^t y(\tau)d\tau = 3u(t) - 2y(t)$$

$$\Downarrow$$

$$\frac{Y(s)}{s} = \frac{3}{s} - 2Y(s) \quad \Rightarrow \quad Y(s) = \frac{3}{2(s+0.5)}$$

Problem Assignments

- Problems: 3.2, 3.3, 3.6. 3.7
- Try the Matlab code in the example in Chapter 3
- Partial Solutions available from the student section of the textbook web site