

Signals and Systems - Chapter 3

The Laplace Transform -Part 2

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The Two-Sided Laplace Transform

The two-sided Laplace transform of a continuous-time function $f(t)$ is

$$F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-st} dt \quad s \in \text{ROC} \quad (3.2)$$

where the variable $s = \sigma + j\Omega$, with Ω as the frequency in rad/sec and σ as a damping factor. ROC stands for the region of convergence—that is, where the integral exists.

The inverse Laplace transform is given by

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds \quad \sigma \in \text{ROC} \quad (3.3)$$

The One-Sided Laplace Transform

The *one-sided Laplace transform* is defined as

$$F(s) = \mathcal{L}[f(t)u(t)] = \int_{0^-}^{\infty} f(t)u(t)e^{-st} dt \quad (3.8)$$

where $f(t)$ is either a causal function or made into a causal function by the multiplication by $u(t)$. The one-sided Laplace transform is of significance given that most of the applications deal with causal systems and signals, and that any signal or system can be decomposed into causal and anti-causal components requiring only the computation of one-sided Laplace transforms.

Inverse Laplace Transform

- Inverting the Laplace transform consists in finding a signal that has the given transform with the given region of convergence (ROC)
- 3 Cases:
 - Inverse of one-sided Laplace transforms giving causal functions
 - Inverse of Laplace transforms with exponentials.
 - Inverse of two-sided Laplace transforms giving anti-causal or noncausal functions

Inverse of One-Sided Laplace Transforms: Partial Fraction Expansion

- Expanding the given function in s into a sum of components of which the inverse Laplace transforms can be found in a table of Laplace transform pairs

$$\boxed{X(s) = \frac{N(s)}{D(s)}} \quad \longrightarrow \quad X(s) = g_0 + g_1 s + \cdots + g_m s^m + \frac{B(s)}{D(s)}$$

\mathcal{L}^{-1}

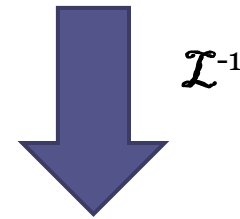
$$x(t) = g_0 \delta(t) + g_1 \frac{d\delta(t)}{dt} + \cdots + g_m \frac{d^m \delta(t)}{dt^m} + \mathcal{L}^{-1} \left[\frac{B(s)}{D(s)} \right]$$

Simple Real Poles

$$X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_k (s - p_k)}$$

$$X(s) = \sum_k \frac{A_k}{s - p_k}$$

$$A_k = X(s)(s - p_k) \Big|_{s=p_k}$$



$$x(t) = \sum_k A_k e^{p_k t} u(t)$$

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s , ROC
1.	$\delta(t)$	1, whole s -plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
4.	$e^{-at}u(t)$, $a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2 + \Omega_0^2}$, $\mathcal{R}e[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2 + \Omega_0^2}$, $\mathcal{R}e[s] > 0$
7.	$e^{-at} \cos(\Omega_0 t)u(t)$, $a > 0$	$\frac{s+a}{(s+a)^2 + \Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at} \sin(\Omega_0 t)u(t)$, $a > 0$	$\frac{\Omega_0}{(s+a)^2 + \Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A e^{-at} \cos(\Omega_0 t + \theta)u(t)$, $a > 0$	$\frac{A \angle \theta}{s+a-j\Omega_0} + \frac{A \angle -\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!} t^{N-1} u(t)$	$\frac{1}{s^N}$, N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!} t^{N-1} e^{-at} u(t)$	$\frac{1}{(s+a)^N}$, N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!} t^{N-1} e^{-at} \cos(\Omega_0 t + \theta)u(t)$	$\frac{A \angle \theta}{(s+a-j\Omega_0)^N} + \frac{A \angle -\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

Simple Real Poles - Example

- Find the causal inverse of:

$$X(s) = \frac{3s + 5}{s^2 + 3s + 2} = \frac{3s + 5}{(s + 1)(s + 2)}$$

$$X(s) = \frac{A_1}{s + 1} + \frac{A_2}{s + 2} \xrightarrow{\mathcal{L}^{-1}} x(t) = [A_1 e^{-t} + A_2 e^{-2t}]u(t) \\ = [2e^{-t} + e^{-2t}]u(t)$$

$$A_1 = X(s)(s + 1)|_{s=-1} = \frac{3s + 5}{s + 2} \Big|_{s=-1} = 2$$

$$A_2 = X(s)(s + 2)|_{s=-2} = \frac{3s + 5}{s + 1} \Big|_{s=-2} = 1$$

Simple Complex Conjugate Poles

$$X(s) = \frac{N(s)}{(s + \alpha)^2 + \Omega_0^2} = \frac{N(s)}{(s + \alpha - j\Omega_0)(s + \alpha + j\Omega_0)}$$



$$X(s) = \frac{A}{s + \alpha - j\Omega_0} + \frac{A^*}{s + \alpha + j\Omega_0}$$

$$A = X(s)(s + \alpha - j\Omega_0)|_{s = -\alpha + j\Omega_0} = |A|e^{j\theta}$$



$$x(t) = 2|A|e^{-\alpha t} \cos(\Omega_0 t + \theta)u(t)$$

Simple Complex Conjugate Poles

- Find the causal inverse Laplace transform of

$$X(s) = \frac{2s + 3}{s^2 + 2s + 4} = \frac{2s + 3}{(s + 1)^2 + 3}$$

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s , ROC
1.	$\delta(t)$	1, whole s -plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
4.	$e^{-at}u(t)$, $a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2 + \Omega_0^2}$, $\mathcal{R}e[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2 + \Omega_0^2}$, $\mathcal{R}e[s] > 0$
7.	$e^{-at} \cos(\Omega_0 t)u(t)$, $a > 0$	$\frac{s+a}{(s+a)^2 + \Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at} \sin(\Omega_0 t)u(t)$, $a > 0$	$\frac{\Omega_0}{(s+a)^2 + \Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A e^{-at} \cos(\Omega_0 t + \theta)u(t)$, $a > 0$	$\frac{A \angle \theta}{s+a-j\Omega_0} + \frac{A \angle -\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!} t^{N-1}u(t)$	$\frac{1}{s^N}$, N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!} t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$, N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!} t^{N-1}e^{-at} \cos(\Omega_0 t + \theta)u(t)$	$\frac{A \angle \theta}{(s+a-j\Omega_0)^N} + \frac{A \angle -\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

$$X(s) = \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{(s+1)^2 + 3} + 2 \frac{s+1}{(s+1)^2 + 3}$$



$$x(t) = \left[\frac{1}{\sqrt{3}} \sin(\sqrt{3}t) + 2 \cos(\sqrt{3}t) \right] e^{-t} u(t)$$

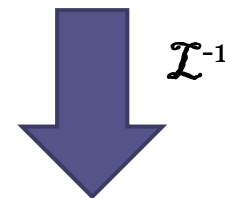
Double Real Poles

$$X(s) = \frac{N(s)}{(s + \alpha)^2} = \frac{a + b(s + \alpha)}{(s + \alpha)^2} = \frac{a}{(s + \alpha)^2} + \frac{b}{s + \alpha}$$

$$a = X(s)(s + \alpha)^2 \Big|_{s=-\alpha}$$

Table 3.1 One-Sided Laplace Transforms

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3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
4.	$e^{-at}u(t)$, $a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2 + \Omega_0^2}$, $\mathcal{R}e[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2 + \Omega_0^2}$, $\mathcal{R}e[s] > 0$
7.	$e^{-at} \cos(\Omega_0 t)u(t)$, $a > 0$	$\frac{s+a}{(s+a)^2 + \Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at} \sin(\Omega_0 t)u(t)$, $a > 0$	$\frac{\Omega_0}{(s+a)^2 + \Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A e^{-at} \cos(\Omega_0 t + \theta)u(t)$, $a > 0$	$\frac{A \angle \theta}{s+a-j\Omega_0} + \frac{A \angle -\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!} t^{N-1} u(t)$	$\frac{1}{s^N}$, N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!} t^{N-1} e^{-at} u(t)$	$\frac{1}{(s+a)^N}$, N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!} t^{N-1} e^{-at} \cos(\Omega_0 t + \theta)u(t)$	$\frac{A \angle \theta}{(s+a-j\Omega_0)^N} + \frac{A \angle -\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$



$$x(t) = [ate^{-\alpha t} + be^{-\alpha t}]u(t)$$

Double Real Poles - Example

- Find the causal inverse Laplace transform of:

$$X(s) = \frac{4}{s(s+2)^2}$$


$$X(s) = \frac{A}{s} + \frac{a + b(s+2)}{(s+2)^2}$$

$$A = X(s)s|_{s=0} = 1$$

$$X(s) - \frac{1}{s} = \frac{4 - (s+2)^2}{s(s+2)^2} = \frac{-(s+4)}{(s+2)^2}$$

$$= \frac{a + b(s+2)}{(s+2)^2}$$

$$\Rightarrow b = -1 \quad a = -2$$


$$x(t) = [1 - 2te^{-2t} - e^{-2t}]u(t)$$

Analysis of LTI Systems

- Complete response for system described by

$$y^{(N)}(t) + \sum_{k=0}^{N-1} a_k y^{(k)}(t) = \sum_{\ell=0}^M b_\ell x^{(\ell)}(t) \quad N > M$$

with initial conditions: $\{y^{(k)}(t), 0 \leq k \leq N - 1\}$

is obtained by inverting the Laplace transform:

$$Y(s) = \frac{B(s)}{A(s)}X(s) + \frac{1}{A(s)}I(s)$$

$$A(s) = \sum_{k=0}^N a_k s^k \quad a_N = 1$$

$$B(s) = \sum_{\ell=0}^M b_\ell s^\ell$$

$$I(s) = \sum_{k=1}^N a_k \left(\sum_{m=0}^{k-1} s^{k-m-1} y^{(m)}(0) \right)$$

Analysis of LTI Systems

- Let $H(s) = \frac{B(s)}{A(s)}$ and $H_1(s) = \frac{1}{A(s)}$

$$Y(s) = H(s)X(s) + H_1(s)I(s) \quad \longrightarrow \quad y(t) = \gamma_{zs}(t) + \gamma_{zi}(t)$$

zero-state response: $\gamma_{zs}(t) = \mathcal{L}^{-1}[H(s)X(s)]$

zero-input response: $\gamma_{zi}(t) = \mathcal{L}^{-1}[H_1(s)I(s)]$

Analysis of LTI Systems - Example

- Find the impulse response $h(t)$ and unit step response $s(t)$ of the system defined by the differential equation:


$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$


- Assume zero initial conditions (LTI)

$$Y(s)[s^2 + 3s + 2] = X(s)$$

$$H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \quad S(s) = \frac{H(s)}{s}$$

$A = 1$ and $B = -1$


$$h(t) = [e^{-t} - e^{-2t}]u(t)$$


$$s(t) = 0.5u(t) - e^{-t}u(t) + 0.5e^{-2t}u(t)$$

Analysis of LTI Systems - Example

- Consider the same system but with nonzero initial conditions

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t) \quad y(0) = 1 \text{ and } dy(t)/dt|_{t=0} = 0$$

- Can we compute $h(t)$??

$$[s^2Y(s) - sy(0) - \frac{dy(t)}{dt}|_{t=0}] + 3[sY(s) - y(0)] + 2Y(s) = X(s)$$

$$Y(s)(s^2 + 3s + 2) - (s + 3) = X(s)$$



Unable to find $H(s)=Y(s)/X(s)$!

$$Y(s) = \frac{X(s)}{(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)}$$

Computation of the Convolution Integral

The Laplace transform of the convolution $y(t) = [x * h](t)$ is given by the product

$$Y(s) = X(s)H(s)$$

where $X(s) = \mathcal{L}[x(t)]$ and $H(s) = \mathcal{L}[h(t)]$. The transfer function of the system $H(s)$ is defined as

$$H(s) = \mathcal{L}[h(t)] = \frac{Y(s)}{X(s)}$$

- $H(s)$ transfers the Laplace transform $X(s)$ of the input into the Laplace transform of the output $Y(s)$.
- Once $Y(s)$ is found, $y(t)$ is computed by means of the inverse Laplace transform.

Computation of the Convolution Integral - Example

Use the Laplace transform to find the convolution $y(t) = [x * h](t)$ when

- (1) the input is $x(t) = u(t)$ and the impulse response is a pulse $h(t) = u(t) - u(t - 1)$, and
- (2) the input and the impulse response of the system are $x(t) = h(t) = u(t) - u(t - 1)$.

The Laplace transforms are $X(s) = \mathcal{L}[u(t)] = 1/s$ and $H(s) = \mathcal{L}[h(t)] = (1 - e^{-s})/s$, so that

$$Y(s) = H(s)X(s) = \frac{1 - e^{-s}}{s^2} \xrightarrow{\mathcal{L}^{-1}} y(t) = r(t) - r(t - 1)$$

In the second case, $X(s) = H(s) = \mathcal{L}[u(t) - u(t - 1)] = (1 - e^{-s})/s$, so that

$$Y(s) = H(s)X(s) = \frac{(1 - e^{-s})^2}{s^2} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2} \xrightarrow{\mathcal{L}^{-1}} y(t) = r(t) - 2r(t - 1) + r(t - 2)$$

Problem Assignments

- Problems: 3.9, 3.10, 3.11, 3.12, 3.13, 3.18, 3.32
- Try the Matlab code in the example in Chapter 3
- Partial Solutions available from the student section of the textbook web site