SIMULATION SYSTEMS

REJECTION METHOD

Prof. Yasser Mostafa Kadah

Rejection Method

- Powerful, general technique for generating random samples whose distribution function p(x)dx (probability of a value occurring between x and x+dx) is known and computable.
- Does not require that the cumulative distribution function (indefinite integral of p(x)) be readily computable
  - Also inverse of that function is not required — which was required for the transformation method

Rejection Method

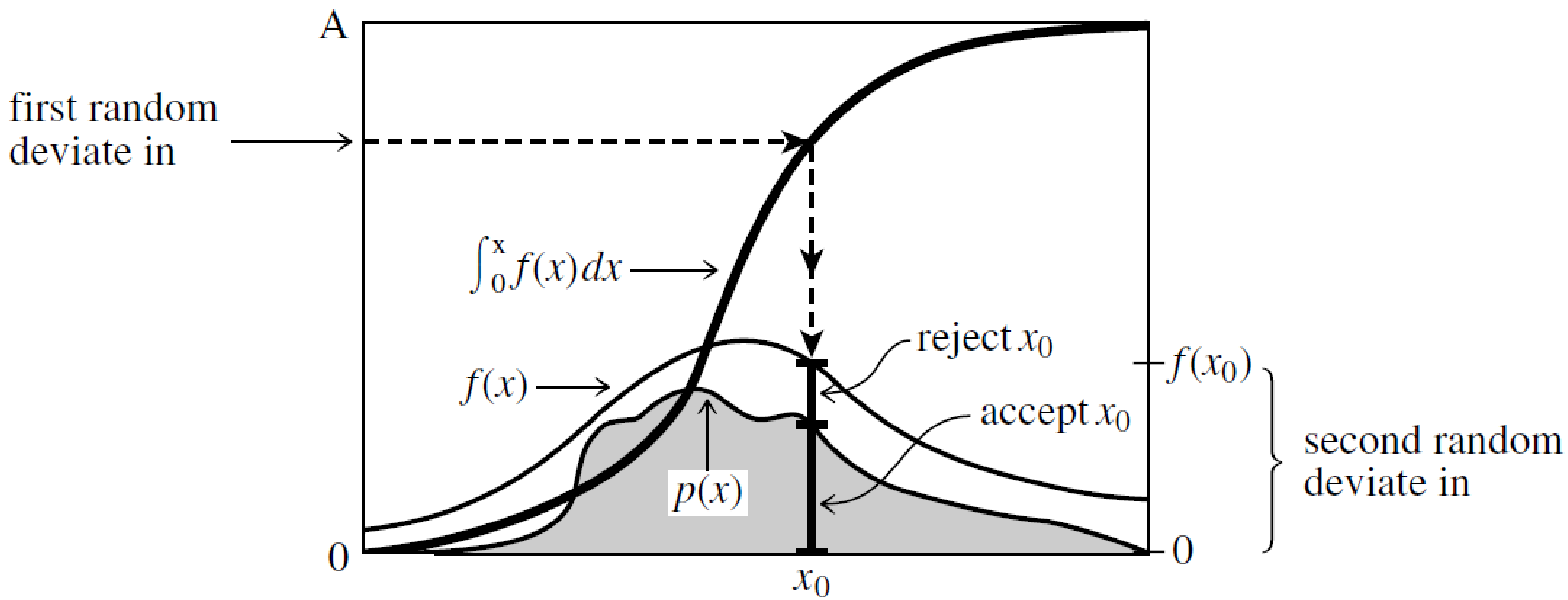
- More practical alternative to the transformation method that we can use to generate random variables from a distribution f(x)
- To perform the rejection method, we need to find a invertible function g(x) that strictly upper bounds f(x):

$$g(x) \geq f(x) \quad \forall x.$$

- Require that we can compute area under g(x):

$$A = \int_{-\infty}^{\infty} g(u)\, du$$

Rejection Method: Idea



- Denote desired probability as p(x) and comparison function (upper bound) be f(x) in the graph above

Rejection Method: Steps

- Rejection method for generating a random deviate x from a known probability distribution p(x) that is everywhere less than some other function f(x):
  - Transformation method is first used to generate a random deviate x of the distribution f
  - A second uniform deviate is used to decide whether to accept or reject that x. If it is rejected, a new deviate of f is found, and so on
  - Ratio of accepted to rejected points is the ratio of the area under p to the area between p and f

Rejection Method: Notes

- Rejection method for some given p(x) requires that one finds some reasonably good comparison function f(x)
  - Function whose indefinite integral is known analytically
  - Analytically invertible to allow transformation method
- Each sample generated requires two uniform random deviates
  - One evaluation of f(x) (to get the coordinate y)
  - One evaluation of p (to decide whether to accept or reject the point x; y)
  - Process may need to be repeated, on the average, **A** times before the final deviate is obtained (losses due to rejection)

Rejection Method: Example

- Required: Sampling from normal distribution
  - x~N(0,1)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- Symmetric about x=0
  - Can sample from positive values of x and then pick a single random bit to decide whether to use +x or -x.
- Use a comparison function that bounds f (x) for positive x

$$g(x) = Ce^{-x/2}$$

  - Select value of C such that this function upper bounds f(x):

$$Ce^{-x/2} \geq \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Rejection Method: Example

- Selecting value of C:

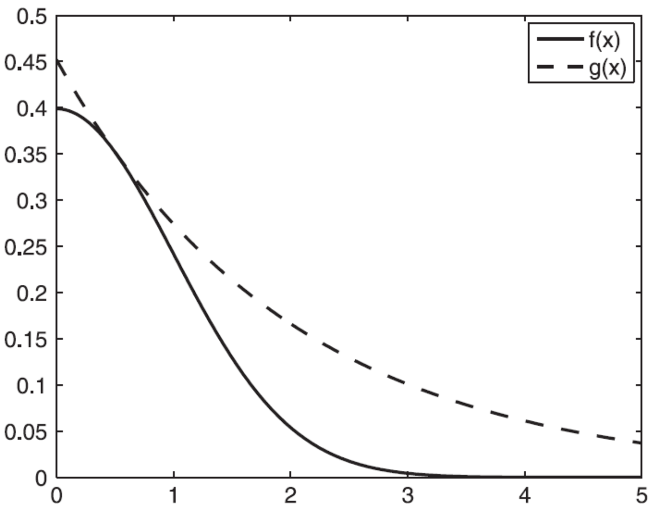
$$Ce^{-x/2} \geq \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \Rightarrow \quad C \geq \frac{1}{\sqrt{2\pi}} e^{(x-x^2)/2}$$

  - C should be selected as the maximum of right hand side

$$\frac{d}{dx}(x - x^2) = 1 - 2x = 0 \Rightarrow x = \frac{1}{2} \quad \Rightarrow \quad C \geq \frac{1}{\sqrt{2\pi}} e^{(1/2-1/4)/2} = \frac{e^{1/8}}{\sqrt{2\pi}}$$

$$\Rightarrow \quad \boxed{g(x) = \frac{e^{1/8}}{\sqrt{2\pi}} e^{-x/2}}$$

$$A = \int_0^\infty \frac{e^{1/8}}{\sqrt{2\pi}} e^{-u/2}\, du = \frac{2e^{1/8}}{\sqrt{2\pi}}$$



Rejection Method: Example

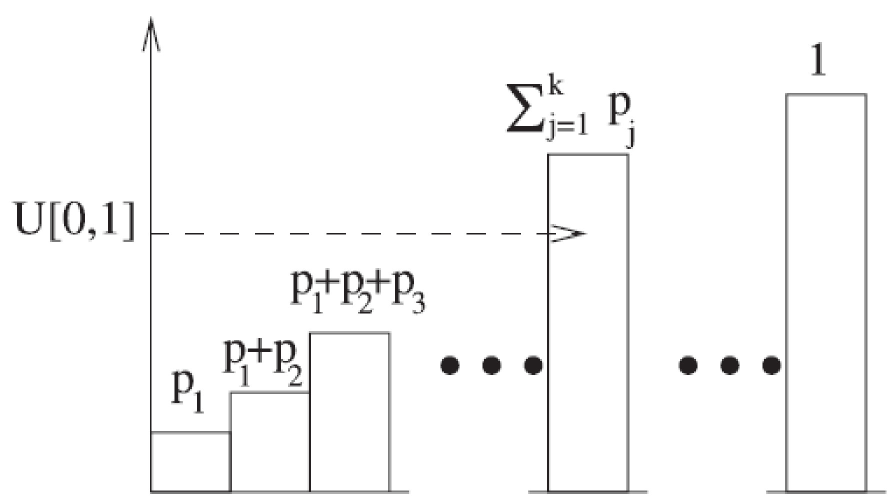
1. Sample $x = Exp\left(\frac{1}{2}\right)$.
2. Sample $y = U\left[0, \frac{e^{1/8}}{\sqrt{2\pi}} e^{-x/2}\right]$.
3. If $y \geq \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, return to step 1.
4. If $I\left(\frac{1}{2}\right) = 1$, return x, else return $-x$.

- Probability of accepting any point will be the ratio of the areas under the two curves.
  - Area under f(x) is ½ since it covers the right half of the normal distribution function.
  - Area under g(x) is A
- Probability of accepting a point = **1/2A ≈ 0.553**
  - Need to try about two points for 1 sample from N(0,1)

Sampling from Discrete Distributions

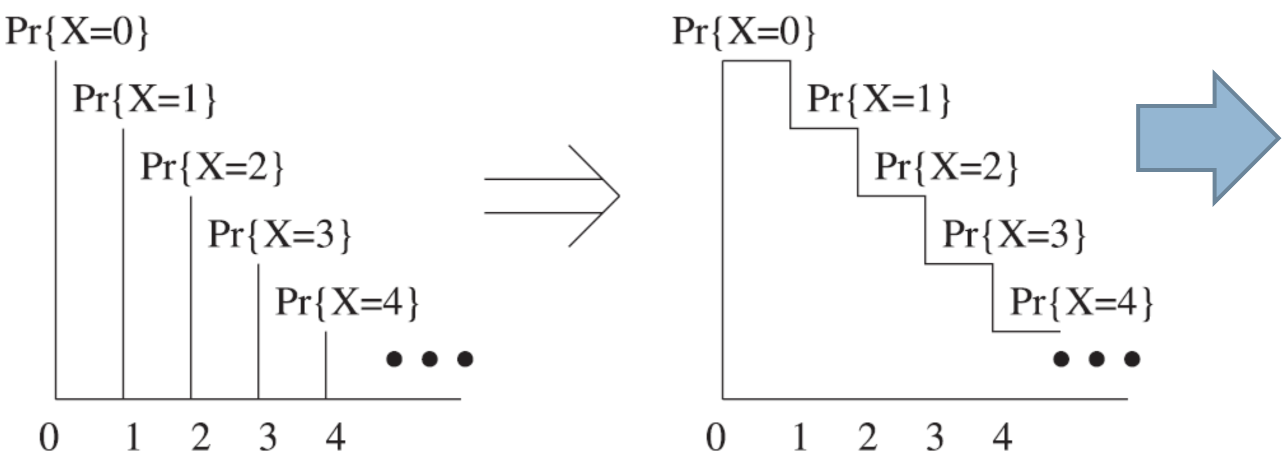
- Sampling from discrete distributions is generally much easier than sampling from continuous distributions.
  - If we have a small finite set of outcomes 1; . . . ; k with probabilities $p_1; \ldots ; p_k$, then we can sample from the distribution implied by those probabilities as follows:

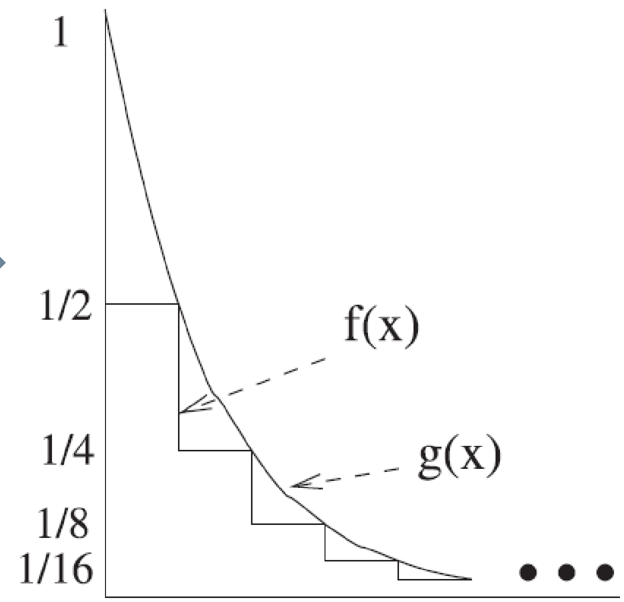
1. Sample $x = U[0, 1]$
2. $i \leftarrow 1$
3. while $(x > p_i)$
   A. $x \leftarrow x - p_i$
   B. $i \leftarrow i + 1$
4. end while
5. return(i)



Sampling from Discrete Distributions

- In cases where discrete transformation method is not practical, we can create a discrete version of the rejection method



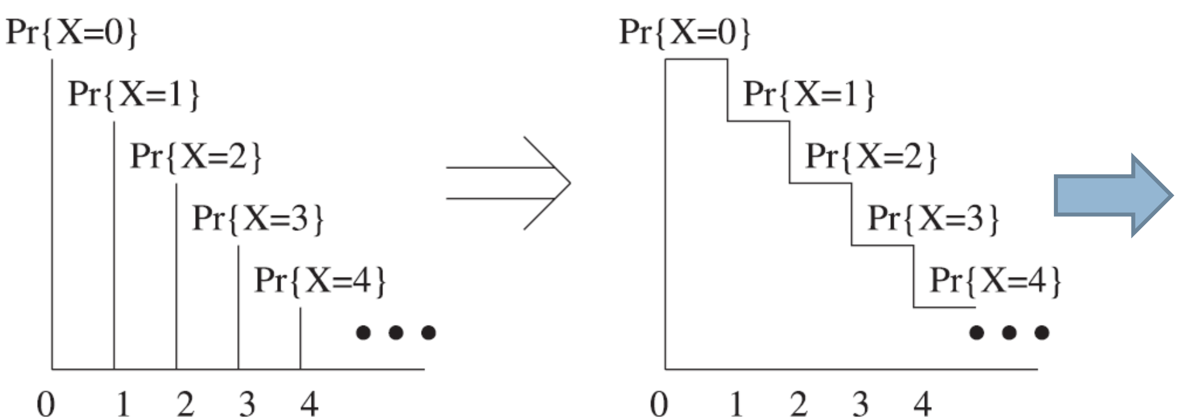


Sampling from Discrete Distributions: Example

- Sampling from a Geom(½) geometric variable.
  - This random variable has the density function:

$$Pr\{X = k\} = \left(\frac{1}{2}\right)^{k-1}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{k}$$

- First convert this discrete probability function into a continuous step function density f(x):



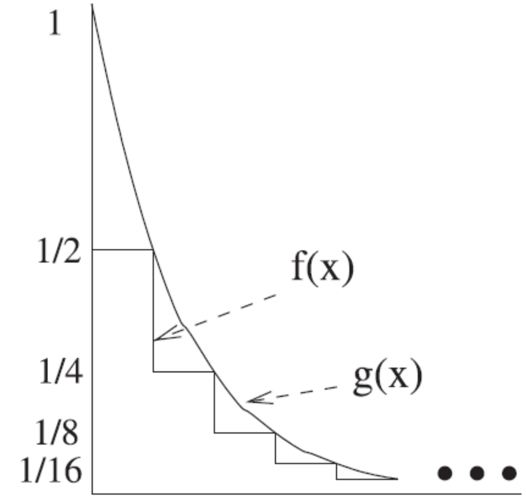
$$f(x) = \begin{cases} 0 & x \le 0 \\ 1/2 & 0 < x \le 1 \\ 1/4 & 1 < x \le 2 \\ 1/8 & 2 < x \le 3 \\ \vdots & \vdots \end{cases}$$

Sampling from Discrete Distributions: Example

- Then we bound f(x) with a continuous function g(x)

$$g(x) = \begin{cases} 0 & x \le 0 \\ \left(\frac{1}{2}\right)^x = e^{-x \ln 2} & x > 0 \end{cases}$$

$$A = \int_{-\infty}^{\infty} g(u)\, du = \int_0^{\infty} e^{-u \ln 2}\, du = \frac{1}{\ln 2}$$



1. Sample X from density $\frac{1}{A} g(x) = \ln 2 e^{-X \ln 2}$ (which is Exp(ln 2)).
2. Sample from $Y = U[0, e^{-X \ln 2}]$.
3. If $Y \le \left(\frac{1}{2}\right)^{\lceil X \rceil}$, then return X, else go to step 1.

Assignments

- Implement a normal distribution N(0,1) sampling method based on the Rejection Method.
- Write a program to sample from the Poisson distribution based on:
  - A) The discrete version of the transformation method
  - B) The discrete version of the rejection method

$$k \sim \text{Poisson}(\lambda), \qquad \lambda > 0$$

$$p(k) = \frac{1}{k!}\lambda^k e^{-\lambda}, \qquad k = 0, 1, \ldots$$