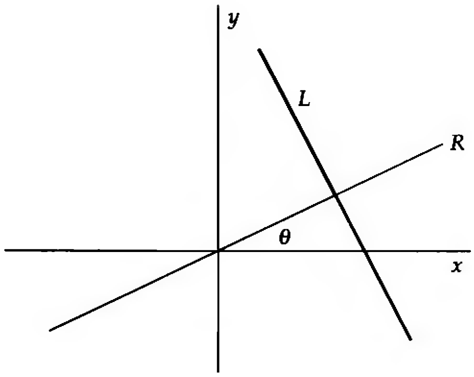


Fig. 13.8 **A density distribution in the (x,y)-plane is scanned by a line $L$ moving in the $R$-direction to generate a projection.**

The factor $\delta(R - x\cos\theta - y\sin\theta)$ is zero everywhere except where its argument is zero, which is along the straight line $x\cos\theta + y\sin\theta = R$ (Fig. 13.8). This straight line $L$ represents the slit when it is at a perpendicular distance $R$ from the origin and inclined at an angle $\theta$ to the $y$-axis. If $\theta$ is kept fixed, say at a value $\theta_1$, while $R$ is varied, then the integral $g_{\theta_1}(R)$ constitutes the projection of the density distribution $f(x,y)$ onto the line $\theta = \theta_1$ as a function of $R$. The resulting profile is referred to as a single scan.

The practical computational method for inversion was arrived at by Fourier transforming the Radon integral equation, finding a method of solution, and then retransforming the steps to end up with data-plane operations in which numerical Fourier transformation is actually dispensed with. The technique, known as modified back-projection (Bracewell and Riddle, 1967), was developed in connection with radioastronomical imaging where a distributed source of radiation is scanned by an antenna that receives from a narrow strip of sky whose orientation $\theta$ can be varied between scans.

The inversion procedure derives from a remarkable connection that exists between the Fourier, Abel, and Hankel transforms and from a generalization known as the Projection-Slice Theorem.

***The Abel-Fourier-Hankel ring of transforms.*** Starting with an even function $f(r)$, if we take the Abel transform, then take the Fourier transform, and finally take the Hankel transform, we return to the original function $f(r)$ as shown in Bracewell (1956). For example, starting with $f(r) = \delta(r - a)$, which is a ring impulse located on the circle $r = a$, we take the Abel transform (Table 13.9) to get $2a/\sqrt{a^2 - x^2}\Pi(x/2a)$, the Fourier transform of which is $2\pi a J_0(2\pi as)$ (Pictorial Dictionary). From Table 13.2 we verify that the Hankel transform of the Bessel function is $\delta(r - a)$, the function we started with.

***Projection-slice theorem.*** When a two-dimensional density distribution is a function of radius alone, all three of the above transformations are one-dimensional but $f(r)$ can be generalized to become a function $f(x,y)$ of both $x$ and $y$ and what was the Hankel transform above generalizes to a function $F(u,v)$ of $u$ and $v$. These two two-dimensional functions constitute a two-dimensional Fourier transform pair, as explained earlier in the chapter in connection with the Hankel transformation. One way of thinking about Fourier transformation in two dimensions is to note that $F(u,0)$, the slice through $F(u,v)$ along the $u$-axis, is given by putting $v = 0$ in the two-dimensional Fourier transform definition to get

$$F(u,0) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)dy\right]e^{-i2\pi ux}dx.$$

The item in square brackets is the projection of $f(x,y)$ on the $x$-axis. The remaining integral with respect to $x$ simply transforms the projection. In consequence, when $f(x, y)$ is given, one slice through $F(u,v)$, namely the one along the $u$-axis, is obtainable by first projecting $f(x,y)$ onto its $x$-axis and then taking a one-dimensional Fourier transform. The Projection-Slice Theorem says that the slice through $F(u,v)$ at any angle $\theta_1$ in the $(u,v)$-plane, i.e., along a line parallel to the axis $R$ in the $(x,y)$-plane, is obtainable as the Fourier transform of the projection of $f(x,y)$ onto the axis $R$ in the $(x,y)$-plane (Bracewell, 1956).

***Reconstruction by modified back projection.*** Now the process of tomography is to project a certain $f(x,y)$ at various angles $\theta$, preferably numerous and equispaced; consequently those parts of the transform $F(u,v)$ can be deduced that lie on slices at corresponding angles. From knowledge of $F(u,v)$ one can recover $f(x,y)$ by two-dimensional Fourier transformation; but to do this one must first interpolate onto a square grid in the $(u,v)$-plane in order to be able to utilize available algorithms. Such numerical interpolation proves to take more time than the transformation. To avoid interpolation we note that in the $(u,v)$-plane, the data points resulting from the various one-dimensional transformations lie on diverging spokes $\theta =$ const. The density of points is thus inversely proportional to radius, a nonuniformity that can be corrected for by multiplication by the absolute value of radius in the $(u,v)$-plane. Let $M$ be a spatial frequency in the $(u,v)$-plane beyond which no content is expected, and let $q = \sqrt{u^2 + v^2}$. Then the correction factor is $\Pi(q/2M) - \Lambda(q/M)$. After such correction a two-dimensional Fourier transform would deliver the desired $f(x,y)$.

But the multiplicative correction to values along the slice in the $(u,v)$-plane corresponds to a rather simple convolution operation on the original projections $g_\theta(R)$ in the data domain, an operation that produces a modified scan

$$\hat{g}_\theta(R) = g_\theta(R) * (2M \operatorname{sinc} 2M R - M \operatorname{sinc}^2 M R).$$

Thus the inversion procedure for the Radon transform is ($a$) to modify each measured scan by simple convolution to get $\hat{g}_\theta(R)$, ($b$) to back-project, and ($c$) to accumulate the separate back projections over the $(x,y)$-plane. Back projection is to distribute the modified scan $\hat{g}_{\theta_1}(R)$ uniformly over the $(x,y)$-plane in the direction perpendicular to the $R$-axis. For more details see Bracewell (1995) and Deans (1983).