Multidimensional Signal Processing Elective Course
Problem Assignment #1 + Solutions

Q1:

For each of the systems below, $x(n)$ is the input and $y(n)$ is the output. Determine which systems are homogeneous, which systems are additive, and which are linear.

(a) $y(n) = \log(x(n))$

(b) $y(n) = 6x(n+2) + 4x(n+1) + 2x(n) + 1$

(c) $y(n) = 6x(n) + [x(n+1)x(n-1)]/x(n)$

(d) $y(n) = x(n)\sin(n\pi/2)$

(e) $y(n) = \text{Re}\{x(n)\}$

(f) $y(n) = \frac{1}{2}[x(n) + x^*(-n)]$

(a) If the system is homogeneous,

$$y(n) = T[cx(n)] = cT[x(n)]$$

for any input $x(n)$ and for all complex constants c. The system $y(n) = \log(x(n))$ is not homogeneous because the response of the system to $x_1(n) = cx(n)$ is

$$y_1(n) = \log(x_1(n)) = \log(cx(n)) = \log c + \log(x(n))$$

which is not equal to $c\log(x(n))$. For the system to be additive, if $y_1(n)$ and $y_2(n)$ are the responses to the inputs $x_1(n)$ and $x_2(n)$, respectively, the response to $x(n) = x_1(n) + x_2(n)$ must be $y(n) = y_1(n) + y_2(n)$. For this system we have

$$T[x_1(n) + x_2(n)] = \log[x_1(n) + x_2(n)] \neq \log[x_1(n)] + \log[x_2(n)]$$

Therefore, the system is not additive. Finally, because the system is neither additive nor homogeneous, the system is nonlinear.

(b) Note that if $y(n)$ is the response to $x(n)$,

$$y(n) = 6x(n+2) + 4x(n+1) + 2x(n) + 1$$

the response to $x_1(n) = cx(n)$ is

$$\begin{aligned} y_1(n) &= 6x_1(n+2) + 4x_1(n+1) + 2x_1(n) + 1 \\ &= c\{6x(n+2) + 4x(n+1) + 2x(n)\} + 1 \end{aligned}$$

However,
$$cy(n) = c\{6x(n+2) + 4x(n+1) + 2x(n) + 1\}$$

which is not the same as $y_1(n)$. Therefore, this system is not homogeneous. Similarly, note that the response to $x(n) = x_1(n) + x_2(n)$ is

$$\begin{aligned} y(n) &= 6x(n+2) + 4x(n+1) + 2x(n) + 1 \\ &= 6\{x_1(n+2) + x_2(n+2)\} + 4\{x_1(n+1) + x_2(n+1)\} + 2\{x_1(n) + x_2(n)\} + 1 \\ &= y_1(n) + y_2(n) - 1 \end{aligned}$$

which is not equal to $y_1(n) + y_2(n)$. Therefore, this system is not additive and, as a result, is nonlinear.

(*c*) This system is homogeneous, because the response of the system to $x_1(n) = cx(n)$ is

$$\begin{aligned} y_1(n) &= 6x_1(n) + \frac{x_1(n+1)x_1(n-1)}{x_1(n)} \\ &= c\left[6x(n) + \frac{x(n+1)x(n-1)}{x(n)}\right] = cy(n) \end{aligned}$$

The system is clearly, however, not additive and therefore is nonlinear.

(*d*) Let $y_1(n)$ and $y_2(n)$ be the responses of the system to the inputs $x_1(n)$ and $x_2(n)$, respectively. The response to the input

$$x(n) = ax_1(n) + bx_2(n)$$

is

$$\begin{aligned} y(n) &= x(n)\sin\left(\frac{n\pi}{2}\right) = [ax_1(n) + bx_2(n)]\sin\left(\frac{n\pi}{2}\right) \\ &= ax_1(n)\sin\left(\frac{n\pi}{2}\right) + bx_2(n)\sin\left(\frac{n\pi}{2}\right) = ay_1(n) + by_2(n) \end{aligned} \tag{1.20}$$

Thus, it follows that this system is linear and, therefore, additive and homogeneous.

(*e*) Because the real part of the sum of two numbers is the sum of the real parts, if $y_1(n)$ is the response of the system to $x_1(n)$, and $y_2(n)$ is the response to $x_2(n)$, the response to $y(n) = y_1(n) + y_2(n)$ is

$$y(n) = \text{Re}\{x_1(n) + x_2(n)\} = \text{Re}\{x_1(n)\} + \text{Re}\{x_2(n)\} = y_1(n) + y_2(n)$$

Therefore the system is additive. It is not homogeneous, however, because

$$\text{Re}\{cx(n)\} \neq c\text{Re}\{x(n)\}$$

unless c is real. Thus, this system is nonlinear.

(*f*) For an input $x(n)$, this system produces an output that is the conjugate symmetric part of $x(n)$. If c is a complex constant, and if the input to the system is $x_1(n) = cx(n)$, the output is

$$y_1(n) = \tfrac{1}{2}\left[x_1(n) + x_1^*(-n)\right] = \tfrac{1}{2}[cx(n) + c^*x^*(-n)] \neq cy(n)$$

Therefore, this system is not homogeneous. This system is, however, additive because

$$\begin{aligned} T[x_1(n) + x_2(n)] &= \tfrac{1}{2}\{[x_1(n) + x_2(n)] + [x_1(-n) + x_2(-n)]^*\} \\ &= \tfrac{1}{2}\{[x_1(n) + x_1^*(-n)] + [x_2(n) + x_2^*(-n)]\} \\ &= T[x_1(n)] + T[x_2(n)] \end{aligned}$$

Q2:

A linear system is one that is both homogeneous and additive.

(a) Give an example of a system that is homogeneous but not additive.

(b) Give an example of a system that is additive but not homogeneous.

There are many different systems that are either homogeneous or additive but not both. One example of a system that is homogeneous but not additive is the following:

$$y(n) = \frac{x(n-1)x(n)}{x(n+1)}$$

Specifically, note that if $x(n)$ is multiplied by a complex constant c, the output will be

$$y(n) = \frac{cx(n-1)\,cx(n)}{cx(n+1)} = c\frac{x(n-1)x(n)}{x(n+1)}$$

which is c times the response to $x(n)$. Therefore, the system is homogeneous. On the other hand, it should be clear that the system is not additive because, in general,

$$\frac{\{x_1(n-1)+x_2(n-1)\}\{x_1(n)+x_2(n)\}}{x_1(n+1)+x_2(n+1)} \neq \frac{x_1(n-1)x_1(n)}{x_1(n+1)} + \frac{x_2(n-1)x_2(n)}{x_2(n+1)}$$

Q3:

Determine whether or not each of the following systems is shift-invariant:

(*a*) $y(n) = x(n) + x(n-1) + x(n-2)$

(*b*) $y(n) = x(n)u(n)$

(*c*) $y(n) = \sum_{k=-\infty}^{n} x(k)$

(*d*) $y(n) = x(n^2)$

(*e*) $y(n) = x((n))_N$ (i.e., $y(n) = x(n \text{ modulo } N)$ as discussed in Prob. 1.8)

(*f*) $y(n) = x(-n)$

(*a*) Let $y(n)$ be the response of the system to an arbitrary input $x(n)$. To test for shift-invariance we want to compare the shifted response $y(n-n_0)$ with the response of the system to the shifted input $x(n-n_0)$. With

$$y(n) = x(n) + x(n-1) + x(n-2)$$

we have, for the shifted response,

$$y(n-n_0) = x(n-n_0) + x(n-n_0-1) + x(n-n_0-2)$$

Now, the response of the system to $x_1(n) = x(n-n_0)$ is

$$\begin{aligned} y_1(n) &= x_1(n) + x_1(n-1) + x_1(n-2) \\ &= x(n-n_0) + x(n-n_0-1) + x(n-n_0-2) \end{aligned}$$

Because $y_1(n) = y(n-n_0)$, the system is shift-invariant.

(*b*) This system is a special case of a more general system that has an input-output description given by

$$y(n) = x(n)f(n)$$

where $f(n)$ is a shift-varying *gain*. Systems of this form are always shift-varying provided $f(n)$ is not a constant. To show this, assume that $f(n)$ is not constant and let n_1 and n_2 be two indices for which $f(n_1) \neq f(n_2)$. With an input $x_1(n) = \delta(n-n_1)$, note that the output $y_1(n)$ is

$$y_1(n) = f(n_1)\delta(n-n_1)$$

If, on the other hand, the input is $x_2(n) = \delta(n-n_2)$, the response is

$$y_2(n) = f(n_2)\delta(n-n_2)$$

Although $x_1(n)$ and $x_2(n)$ differ only by a shift, the responses $y_1(n)$ and $y_2(n)$ differ by a shift and a change in amplitude. Therefore, the system is shift-varying.

(*c*) Let

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

be the response of the system to an arbitrary input $x(n)$. The response of the system to the shifted input $x_1(n) = x(n-n_0)$ is

$$y_1(n) = \sum_{k=-\infty}^{n} x_1(k) = \sum_{k=-\infty}^{n} x(k-n_0) = \sum_{k=-\infty}^{n-n_0} x(k)$$

Because this is equal to $y(n-n_0)$, the system is shift-invariant.

(d) This system is shift-varying, which may be shown with a simple counterexample. Note that if $x(n) = \delta(n)$, the response will be $y(n) = \delta(n)$. However, if $x_1(n) = \delta(n-2)$, the response will be $y_1(n) = x_1(n^2) = \delta(n^2-2) = 0$, which is not equal to $y(n-2)$. Therefore, the system is shift-varying.

(e) With $y(n)$ the response to $x(n)$, note that for the input $x_1(n) = x(n-N)$, the output is

$$y_1(n) = x((n-N))_N = x((n))_N$$

which is the same as the response to $x(n)$. Because $y_1(n) \neq y(n-N)$, in general, this system is not shift-invariant.

(f) This system may easily be shown to be shift-varying with a counterexample. However, suppose we use the direct approach and let $x(n)$ be an input and $y(n) = x(-n)$ be the response. If we consider the shifted input, $x_1(n) = x(n-n_0)$, we find that the response is

$$y_1(n) = x_1(-n) = x(-n-n_0)$$

However, note that if we shift $y(n)$ by n_0,

$$y(n-n_0) = x(-(n-n_0)) = x(-n+n_0)$$

which is not equal to $y_1(n)$. Therefore, the system is shift-varying.

Q4:

A linear discrete-time system is characterized by its response $h_k(n)$ to a delayed unit sample $\delta(n-k)$. For each linear system defined below, determine whether or not the system is shift-invariant.

(*a*) $h_k(n) = (n-k)u(n-k)$

(*b*) $h_k(n) = \delta(2n-k)$

(*c*) $h_k(n) = \begin{cases} \delta(n-k-1) & k \text{ even} \\ 5u(n-k) & k \text{ odd} \end{cases}$

(*a*) Note that $h_k(n)$ is a function of $n-k$. This suggests that the system is shift-invariant. To verify this, let $y(n)$ be the response of the system to $x(n)$:

$$y(n) = \sum_{k=-\infty}^{\infty} h_k(n)x(k)$$

$$= \sum_{k=-\infty}^{\infty} (n-k)u(n-k)x(k) = \sum_{k=-\infty}^{n} (n-k)x(k) \qquad (1.21)$$

The response to a shifted input, $x(n-n_0)$, is

$$y_1(n) = \sum_{k=-\infty}^{\infty} x(k-n_0)h_k(n) = \sum_{k=-\infty}^{\infty} (n-k)u(n-k)x(k-n_0)$$

$$= \sum_{k=-\infty}^{n} (n-k)x(k-n_0)$$

With the substitution $l = k - n_0$ this becomes

$$y_1(n) = \sum_{l=-\infty}^{n-n_0} (n-n_0-l)x(l)$$

From the expression for $y(n)$ given in Eq. (*1.21*), we see that

$$y(n-n_0) = \sum_{k=-\infty}^{n-n_0} (n-n_0-k)x(k)$$

which is the same as $y_1(n)$. Therefore, this system is shift-invariant.

(*b*) For the second system, $h_k(n)$ is *not* a function of $n-k$. Therefore, we should expect this system to be shift-varying. Let us see if we can find an example that demonstrates that it is a shift-varying system. For the input $x(n) = \delta(n)$, the response is

$$y(n) = h_0(n) = \delta(2n) = \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases}$$

If we delay $x(n)$ by 1, the response to $x_1(n) = \delta(n-1)$ is

$$y_1(n) = h_1(n) = \delta(2n-1) = 0$$

Because $y_1(n) \neq y(n-1)$, the system is shift-varying.

(*c*) Finally, for the last system, we see that although $h_k(n)$ is a function of $n-k$ for k even and a function of $(n-k)$ for k odd,

$$h_k(n) \neq h_{k-1}(n-1)$$

In other words, the response of the system to $\delta(n-k-1)$ is not equal to the response of the system to $\delta(n-k)$ delayed by 1. Therefore, this system is shift-varying.

Q5:

If the response of a linear shift-invariant system to a unit step (i.e., the step response) is

$$s(n) = n\left(\tfrac{1}{2}\right)^n u(n)$$

find the unit sample response, $h(n)$.

In this problem, we begin by noting that

$$\delta(n) = u(n) - u(n-1)$$

Therefore, the unit sample response, $h(n)$, is related to the step response, $s(n)$, as follows:

$$h(n) = s(n) - s(n-1)$$

Thus, given $s(n)$, we have

$$\begin{aligned} h(n) &= s(n) - s(n-1) \\ &= n\left(\tfrac{1}{2}\right)^n u(n) - (n-1)\left(\tfrac{1}{2}\right)^{n-1} u(n-1) \\ &= \left[n\left(\tfrac{1}{2}\right)^n - 2(n-1)\left(\tfrac{1}{2}\right)^n\right] u(n-1) \\ &= (2-n)\left(\tfrac{1}{2}\right)^n u(n-1) \end{aligned}$$

Q6:

Given that $x(n)$ is the system input and $y(n)$ is the system output, which of the following systems are causal?

(a) $y(n) = x^2(n)u(n)$

(b) $y(n) = x(|n|)$

(c) $y(n) = x(n) + x(n-3) + x(n-10)$

(d) $y(n) = x(n) - x(n^2 - n)$

(e) $y(n) = \prod_{k=1}^{N} x(n-k)$

(f) $y(n) = \sum_{k=n}^{\infty} x(n-k)$

(a) The system $y(n) = x^2(n)u(n)$ is *memoryless* (i.e., the response of the system at time n depends only on the input at time n and on no other values of the input). Therefore, this system is causal.

(b) The system $y(n) = x(|n|)$ is an example of a noncausal system. This may be seen by looking at the output when $n < 0$. In particular, note that $y(-1) = x(1)$. Therefore, the output of the system at time $n = -1$ depends on the value of the input at a future time.

(c) For this system, in order to compute the output $y(n)$ at time n all we need to know is the value of the input $x(n)$ at times n, $n-3$, and $n-10$. Therefore, this system must be causal.

(d) This system is noncausal, which may be seen by evaluating $y(n)$ for $n < 0$. For example,

$$y(-1) = x(-1) - x(2)$$

Because $y(-1)$ depends on the value of $x(2)$, which occurs after time $n = -1$, this system is noncausal.

(e) The output of this system at time n is the product of the values of the input $x(n)$ at times $n-1, \ldots, n-N$. Therefore, because the output depends only on past values of the input signal, the system is causal.

(f) This system is not causal, which may be seen easily if we rewrite the system definition as follows:

$$y(n) = \sum_{k=n}^{\infty} x(n-k) = \sum_{l=-\infty}^{0} x(l)$$

Therefore, the input must be known for all $n \leq 0$ to determine the output at time n. For example, to find $y(-5)$ we must know $x(0), x(-1), x(-2), \ldots$. Thus, the system is noncausal.

Q7:

Determine which of the following systems are stable:

(*a*) $y(n) = x^2(n)$

(*b*) $y(n) = e^{x(n)}/x(n-1)$

(*c*) $y(n) = \cos(x(n))$

(*d*) $y(n) = \sum_{k=-\infty}^{n} x(k)$

(*e*) $y(n) = \log(1 + |x(n)|)$

(*f*) $y(n) = x(n) * \cos(n\pi/8)$

(*a*) Let $x(n)$ be any bounded input with $|x(n)| < M$. Then it follows that the output, $y(n) = x^2(n)$, may be bounded by

$$|y(n)| = |x(n)|^2 < M^2$$

Therefore, this system is stable.

(*b*) This system is clearly not stable. For example, note that the response of the system to a unit sample $x(n) = \delta(n)$ is infinite for all values of n except $n = 1$.

(*c*) Because $|\cos(x)| \leq 1$ for all x, this system is stable.

(*d*) This system corresponds to a digital integrator and is unstable. Consider, for example, the step response of the system. With $x(n) = u(n)$ we have, for $n \geq 0$,

$$y(n) = \sum_{k=-\infty}^{n} u(k) = (n+1)$$

Although the input is bounded, $|x(n)| \leq 1$, the response of the system is unbounded.

(*e*) This system may be shown to be stable by using the following inequality:

$$\log(1+x) \leq x \qquad x \geq 0$$

Specifically, if $x(n)$ is bounded, $|x(n)| < M$,

$$|y(n)| = |\log(1 + |x(n)|)| \leq 1 + |x(n)| < 1 + M$$

Therefore, the output is bounded, and the system is stable.

(*f*) This system is not stable. This may be seen by considering the bounded input $x(n) = \cos(n\pi/8)$. Specifically, note that the output of the system at time $n = 0$ is

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = \sum_{k=-\infty}^{\infty} \cos\left(\frac{n\pi}{8}\right)\cos\left(-\frac{n\pi}{8}\right) = \sum_{k=-\infty}^{\infty} \cos^2\left(\frac{n\pi}{8}\right)$$

which is unbounded. Alternatively, because the input-output relation is one of convolution, this is a linear shift-invariant system with a unit sample response

$$h(n) = \cos\left(\frac{n\pi}{8}\right)$$

Because a linear shift-invariant system will be stable only if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

we see that this system is not stable.

Q8:

Determine whether or not the signals below are periodic and, for each signal that is periodic, determine the fundamental period.

(*a*) $x(n) = \cos(0.125\pi n)$

(*b*) $x(n) = \text{Re}\{e^{jn\pi/12}\} + \text{Im}\{e^{jn\pi/18}\}$

(*c*) $x(n) = \sin(\pi + 0.2n)$

(*d*) $x(n) = e^{j\frac{\pi}{16}n} \cos(n\pi/17)$

(*a*) Because $0.125\pi = \pi/8$, and

$$\cos\left(\frac{\pi}{8}n\right) = \cos\left(\frac{\pi}{8}(n + 16)\right)$$

$x(n)$ is periodic with period $N = 16$.

(*b*) Here we have the sum of two periodic signals,

$$x(n) = \cos(n\pi/12) + \sin(n\pi/18)$$

with the period of the first signal being equal to $N_1 = 24$, and the period of the second, $N_2 = 36$. Therefore, the period of the sum is

$$N = \frac{N_1 N_2}{\gcd(N_1, N_2)} = \frac{(24)(36)}{\gcd(24, 36)} = \frac{(24)(36)}{12} = 72$$

(*c*) In order for this sequence to be periodic, we must be able to find a value for N such that

$$\sin(\pi + 0.2n) = \sin(\pi + 0.2\,(n + N))$$

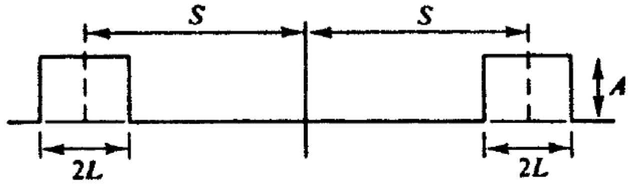
The sine function is periodic with a period of 2π. Therefore, $0.2N$ must be an integer multiple of 2π. However, because π is an irrational number, no integer value of N exists that will make the equality true. Thus, this sequence is aperiodic.

(*d*) Here we have the product of two periodic sequences with periods $N_1 = 32$ and $N_2 = 34$. Therefore, the fundamental period is

$$N = \frac{(32)(34)}{\gcd(32, 34)} = \frac{(32)(34)}{2} = 544$$

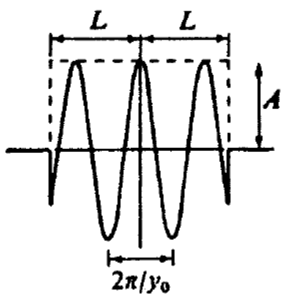
Q9: Derive the Fourier transformation of the following functions:

a.



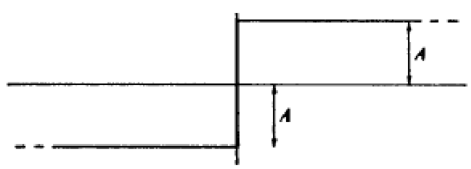
$$f(x) = \begin{cases} A & [(S - L) < |x| < (S + L)] \\ 0 & [\text{otherwise}] \end{cases}$$

b.



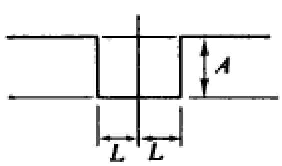
$$f(x) = \begin{cases} A \cos y_0 x & [|x| < L] \\ 0 & [|x| > L] \end{cases}$$

c.



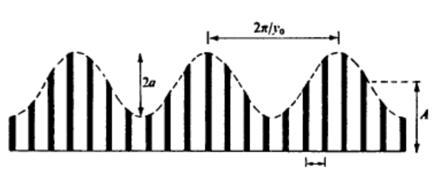
$$\begin{cases} +A & [x > 0] \\ -A & [x < 0] \end{cases} \quad [f(x) = A\,\mathrm{sgn}(x)]$$

d.



$$f(x) = \begin{cases} A & [|x| > L] \\ 0 & [|x| < L] \end{cases}$$

e.

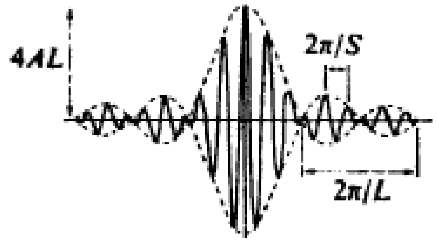


$$\sum \delta(x - nx_0)\{A + a \cos y_0 x\} \quad (1)$$

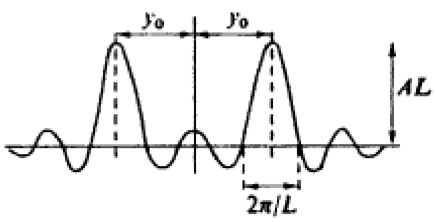
$$\sum \delta(x - nx_0)\{A + a \sin y_0 x\} \quad (2)$$

$$[n = 0, \pm 1, \pm 2, \ldots]$$

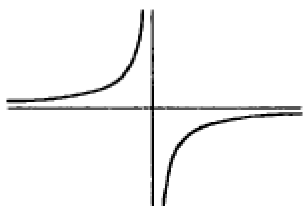
Q9 Answers:



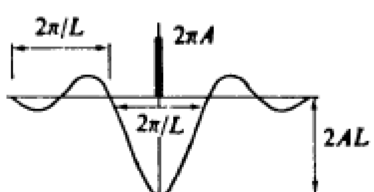
$$4A\,\frac{\cos Sy \sin Ly}{y}$$



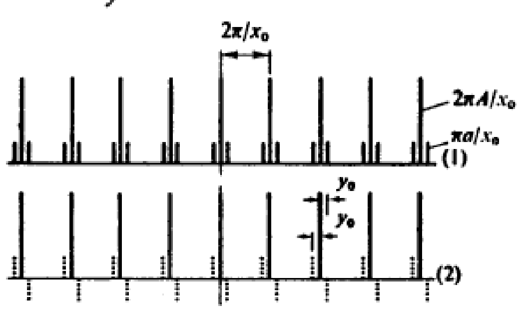
$$A\left[\frac{\sin L(y-y_0)}{(y-y_0)}+\frac{\sin L(y+y_0)}{(y+y_0)}\right]$$



$$-2iA\,\frac{1}{y}$$



$$2\pi A\delta(y)-2A\,\frac{\sin Ly}{y}$$



$$\sum_n \frac{2\pi}{x_0}\left\{A\delta\left(y-n\frac{2\pi}{x_0}\right)+\frac{a}{2}\,\delta\left(y-n\frac{2\pi}{x_0}+y_0\right)+\frac{a}{2}\,\delta\left(y-\frac{2\pi}{x_0}-y_0\right)\right\} \qquad (1)$$

$$\sum_n \frac{2\pi}{x_0}\left\{A\delta\left(y-n\frac{2\pi}{x_0}\right)+\frac{ia}{2}\,\delta\left(y-n\frac{2\pi}{x_0}+y_0\right)-\frac{ia}{2}\,\delta\left(y-n\frac{2\pi}{x_0}-y_0\right)\right\} \qquad (2)$$

$$[n=0,\pm 1,\pm 2,\ldots] \qquad (3.37)$$