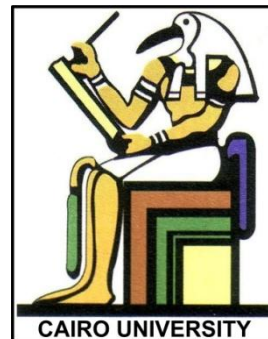


Medical Equipment I - 2009

Chapter 4

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Web: <http://ymk.k-space.org/courses.htm>



Transport in an Infinite Medium

- Definitions
- Continuity equations
- Brownian motion
- Motion in a gas
- Motion in a liquid
- Diffusion
- Applications

Definitions

- Flow rate, volume flux or volume current (i)
 - Total volume of material transported per unit time
 - Units: m^3s^{-1}
- Mass flux
- Particle flux

Definitions

■ Particle fluence

- Number of particles transported per unit area across an imaginary surface
- Units: m^{-2}

■ Volume fluence

- Volume transported per unit area across an imaginary surface
- Units: $m^3m^{-2} = m$

Definitions

- Fluence rate or flux density
 - Amount of “something” transported across an imaginary surface per unit area per unit time
 - Vector pointing in the direction the “something” moves and is denoted by \mathbf{j}
 - Units: “something” $\text{m}^{-2}\text{s}^{-1}$
 - Subscript to denote what “something” is

Definitions

TABLE 4.1. Units and names for j and jS in various fields.

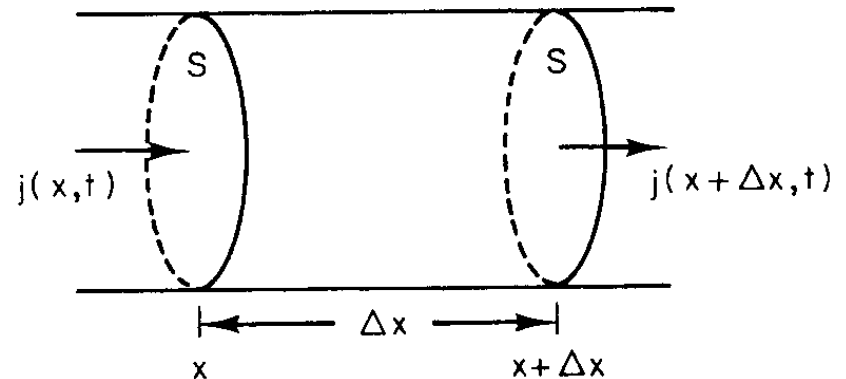
	j		jS	
	Units	Names	Units	Names
Particles	$\text{m}^{-2} \text{s}^{-1}$	Particle fluence rate Particle current density Particle flux density Particle flux	s^{-1}	Particle flux Particle current Particle flux
Electric charge	$\text{C m}^{-2} \text{s}^{-1}$ or A m^{-2}	Current density	C s^{-1} or A	Current
Mass	$\text{kg m}^{-2} \text{s}^{-1}$	Mass fluence rate Mass flux density Mass flux	kg s^{-1}	Mass flux Mass flow
Energy	$\text{J m}^{-2} \text{s}^{-1}$ or W m^{-2}	Energy fluence rate Intensity Energy flux	J s^{-1} or W	Energy flux Power

[Continuity Equation: 1D]

- We deal with substances that do not “appear” or “disappear”
 - Conserved
- Conservation of mass leads to the derivation of the continuity equation

[Continuity Equation: 1D]

- Consider the case of a number of particles
 - Fluence rate: j particles/unit area/unit time
- Value of j may depend on position in tube and time
 - $j = j(x, t)$
- Let volume of particles in the volume shown to be $N(x, t)$
 - Change after $\Delta t = \Delta N$



Continuity Equation: 1D

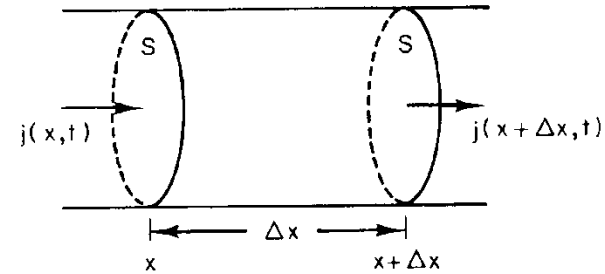
$$\Delta N = [j(x, t) - j(x + \Delta x, t)] S \Delta t$$

- As $\Delta x \rightarrow 0$,

$$j(x, t) - j(x + \Delta x, t) = -\frac{\partial j(x, t)}{\partial x} \Delta x$$

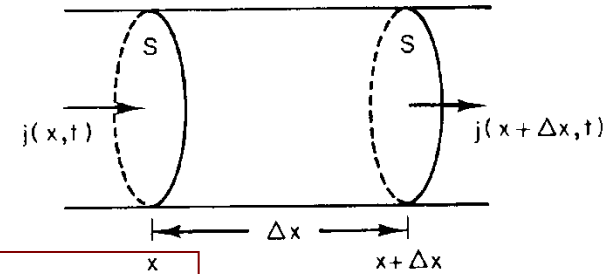
- Similarly, increase in $N(x, t)$ is,

$$\Delta N(x, t) = N(x, t + \Delta t) - N(x, t) = \frac{\partial N}{\partial t} \Delta t$$



Continuity Equation: 1D

- Hence,



$$\frac{\partial}{\partial t} N(x, t) = -(S\Delta x) \frac{\partial}{\partial x} j(x, t)$$

- Then, the continuity equation in 1D is,

$$\frac{\partial C}{\partial t} = - \frac{\partial j}{\partial x}$$

[Solvent Drag (Drift)]

- One simple way that solute particles can move is to drift with constant velocity.
 - Carried along by the solvent,
- Process called *drift or solvent drag*.

$$\mathbf{j}_s = C \mathbf{j}_v$$

Brownian Motion

- Application of thermal equilibrium at temperature T
- Kinetic energy in 1D = $k_B T / 2$
- Kinetic energy in 3D = $3k_B T / 2$
- Random motion \rightarrow mean velocity $\bar{v} = 0$
 - can only deal with mean-square velocity $\overline{v^2}$

$$\frac{1}{2} m \overline{v^2} = \frac{3k_B T}{2} \Rightarrow v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\frac{3k_B T}{m}}$$

Brownian Motion

TABLE 4.2. Values of the rms velocity for various particles at body temperature.

Particle	Molecular weight	Mass (kg)	v_{rms} (m s ⁻¹)
H ₂	2	3.4×10^{-27}	1940
H ₂ O	18	3×10^{-26}	652
O ₂	32	5.4×10^{-26}	487
Glucose	180	3×10^{-25}	200
Hemoglobin	65 000	1×10^{-22}	11
Bacteriophage	6.2×10^6	1×10^{-20}	1.1
Tobacco mosaic virus	40×10^6	6.7×10^{-20}	0.4
<i>E. coli</i>		2×10^{-15}	0.0025

[Motion in a Gas]

- Brownian motion of particles: collisions
- Mean Free Path
 - Average distance between successive collisions
- Collision Time
 - Average time between successive collisions

Motion in a Gas

- Consider $N(x)$ to be number of particles without collision after a distance x
- For short distances dx , probability of collision is proportional to dx

$$dN = N(x) \left(\frac{1}{\lambda} \right) dx \quad \rightarrow \quad N(x) = N_0 e^{-x/\lambda}$$

Motion in a Gas

- Average distance = mean free path

$$\bar{x} = \frac{1}{N_0} \int_0^{\infty} x \frac{N(x)}{\lambda} dx = -\lambda \left[e^{-x/\lambda} \left(\frac{x}{\lambda} + 1 \right) \right]_0^{\infty} = \lambda$$

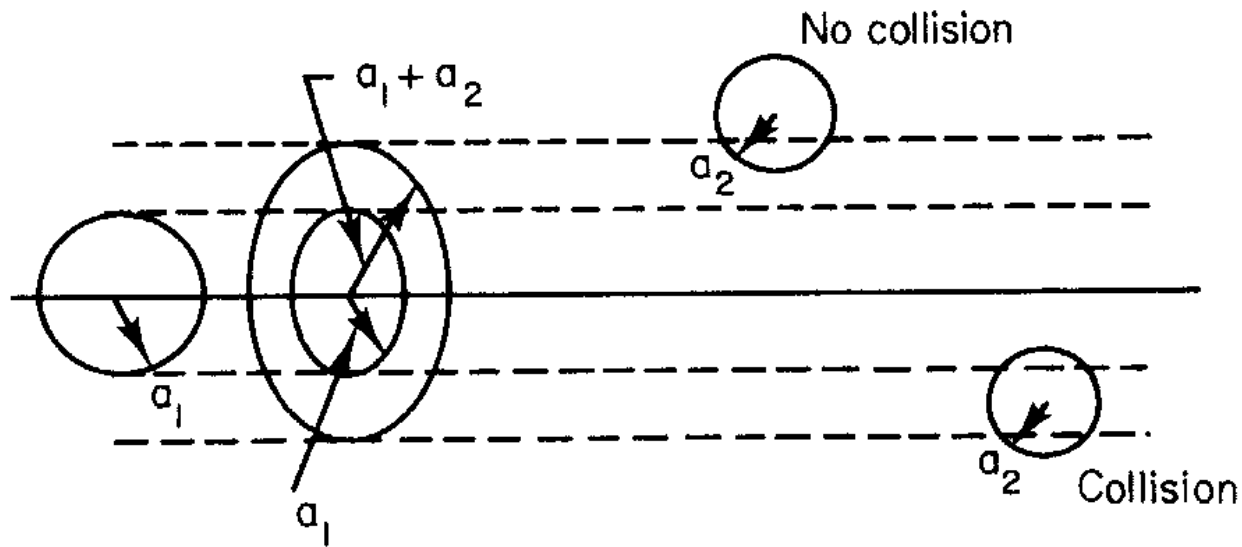
- Similar argument can be made for time

$$N(t) = N_0 e^{-t/t_c}$$

- Collision time = t_c

Motion in a Gas

- Need to evaluate λ and t_c
- Consider one particle moving with a radius a_1
- Consider stationary particles with radius a_2



- Calculate collision possibilities

Motion in a Gas

- After moving a distance x , volume covered is given by,

$$V(x) = \pi(a_1 + a_2)^2 x$$

- On average, when a particle travels mean free path, there is one collision
 - Average number of particles in $V(\lambda)=1$
 - Concentration = $1/V(\lambda)$

$$C = 1/V(\lambda) = 1/\pi(a_1 + a_2)^2 \lambda \Rightarrow \lambda = \frac{1}{\pi(a_1 + a_2)^2 C}$$

Motion in a Gas

- Collision *Cross Section* is $\pi(a_1 + a_2)^2$
 - Important for radiation interaction
- Example: gas at STP, volume of 1 mol = 22.4 L ($C = 2.7 \times 10^{25} \text{m}^{-3}$), $a_1 = a_2 = 0.15 \text{ nm}$
 - $\lambda = 0.13 \text{ } \mu\text{m}$
 - 1000 times the molecular size
 - Assumption of infrequent collisions justified

Motion in a Gas

- Given mean free path λ ,

$$t_c = \frac{\lambda}{\bar{v}}$$

- Taking the average speed as v_{rms} ,

$$t_c \approx \lambda \left(\frac{m}{3k_B T} \right)^{1/2}$$

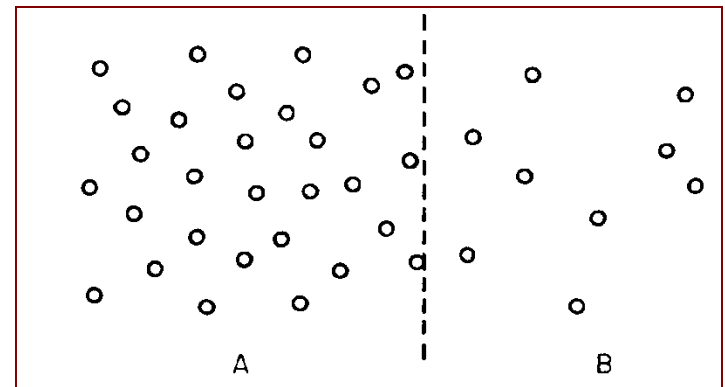
- Dependence on $m^{1/2}$ and λ
- For air and room temperature, $t_c = 2 \times 10^{-10} \text{s}$

Motion in a Liquid

- Direct substitution in Gas equations?
- For water,
 - $\lambda = a = 0.1 \text{ nm}$ → assumption broken
 - $t_c \sim 10^{-13} \text{ s}$ → much more frequent
 - Wrong calculations
 - However, concept appears to be valid!

Diffusion: Fick's First Law

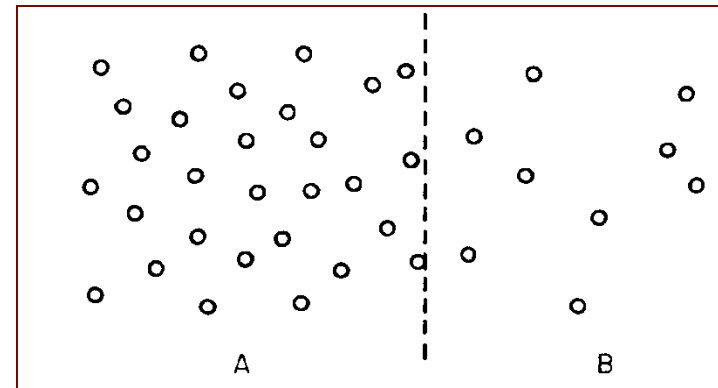
- Diffusion: random movement of particles from a region of higher concentration to a region of lower concentration
- Diffusing particles move independently
- Solvent at rest
 - Solute transport



Diffusion: Fick's First Law

- If solute concentration is uniform, no net flow
- If solute concentration is different, net flow occurs

$$j_x = -D \frac{\partial C}{\partial x}$$



D: Diffusion constant (m^2s^{-1})

Diffusion: Fick's First Law

TABLE 4.3. Various forms of the transport equation.

Substance	Equation	Units of j	Units of the constant
flowing			
Particles	$j_s = -D \frac{\partial C}{\partial x}$	$\text{m}^{-2} \text{s}^{-1}$	$\text{m}^2 \text{s}^{-1}$
Mass	$j_m = -D \frac{\partial C}{\partial x}$	$\text{kg m}^{-2} \text{s}^{-1}$	$\text{m}^2 \text{s}^{-1}$
Heat	$j_H = -\kappa \frac{\partial T}{\partial x}$	$\text{J m}^{-2} \text{s}^{-1}$ or kg s^{-3}	$\text{J K}^{-1} \text{m}^{-1} \text{s}^{-1}$
Electric charge	$j_e = -\sigma \frac{\partial V}{\partial x}$	$\text{C m}^{-2} \text{s}^{-1}$	$\text{C m}^{-1} \text{s}^{-1} \text{V}^{-1}$ or $\Omega^{-1} \text{m}^{-1}$
Viscosity (y component of momentum transported in the x direction)	$\frac{F}{S} = -\eta \frac{\partial v_y}{\partial x}$	N m^{-2} or $\text{kg m}^{-1} \text{s}^{-2}$	$\text{kg m}^{-1} \text{s}^{-1}$ or Pa s

Diffusion: Fick's Second Law

- Consider 1D case
- Fick's first law

$$j_x = -D \frac{\partial C}{\partial x}$$

- Continuity equation

$$\frac{\partial C}{\partial t} = -\frac{\partial j}{\partial x}$$

Diffusion: Fick's Second Law

- Combining Fick's first law and continuity equation,

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} \rightarrow \text{Fick's second law}$$

- 3D Case,

$$\frac{\partial C}{\partial t} = D \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} \right)$$

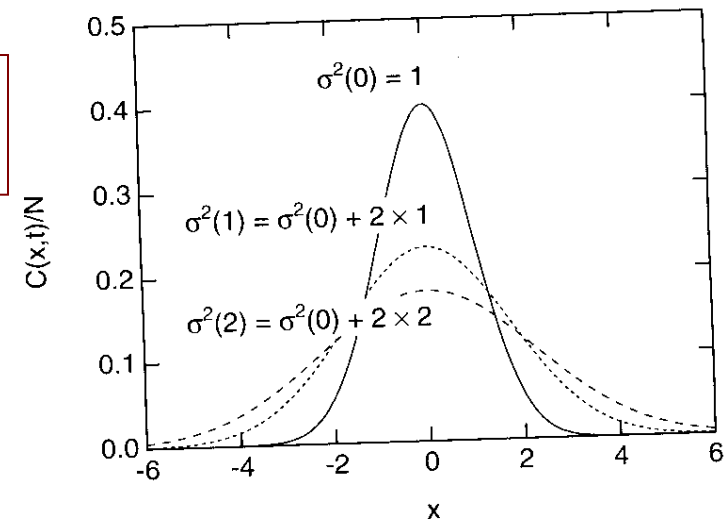
Diffusion: Fick's Second Law

- Solving Fick's second law for $C(x,t)$
 - Substitution

$$C(x,t) = \frac{N}{\sqrt{2\pi}\sigma(t)} e^{-x^2/2\sigma^2(t)}$$

where,

$$\sigma^2(t) = 2Dt + \sigma^2(0)$$



[Applications]

- Kidney dialysis
- Tissue perfusion
- Blood oxygenation in the lung

[Problem Assignments]

- Information posted on web site
- Problems 1, 4, 5, 6, 8, 18, 19