Medical Equipment I - 2009
Chapter 6

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Web: http://ymk.k-space.org/courses.htm

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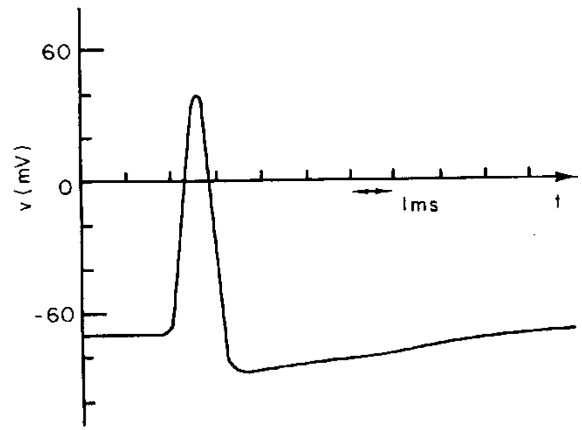
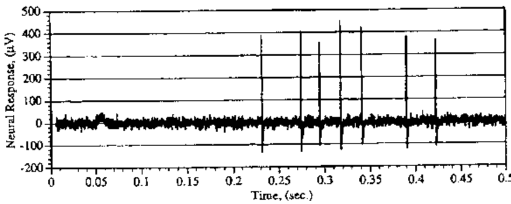
Physiology of Nerve and Muscle Cells

- Action potential
- Transmitted through axon
  - no change in shape
- Myelinated/unmyelinated nerve fibers
  - Nodes of Ranvier
  - Speed of conduction
- Coding using repetition





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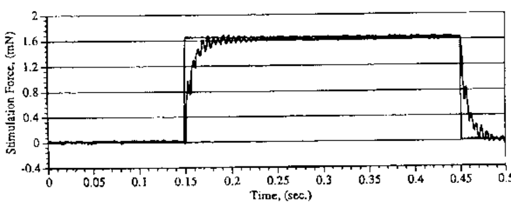
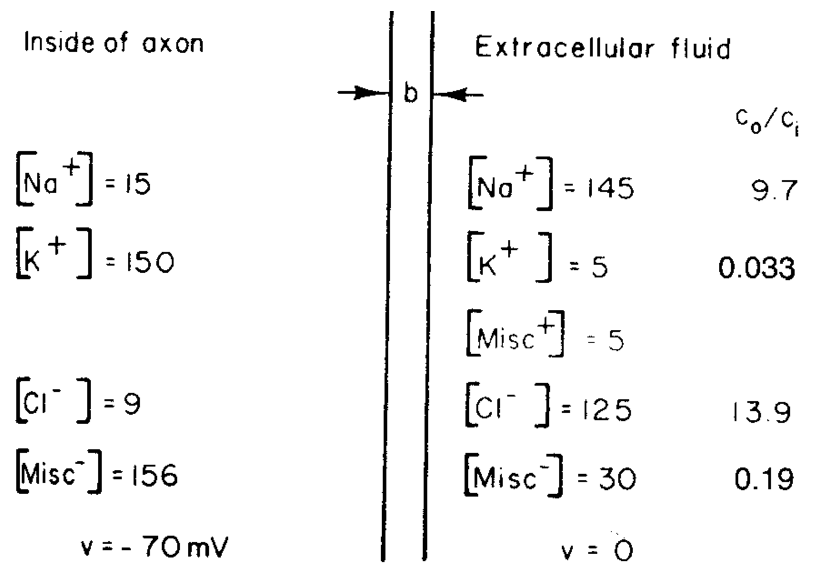


(b)

Physiology of Nerve and Muscle Cells

- Synapse or junction
  - Ach neurotransmitter packets (quanta)
- Extra-/intra- cellular fluids ion concentrations
  - Nernst potential ?
  - Permeability ?



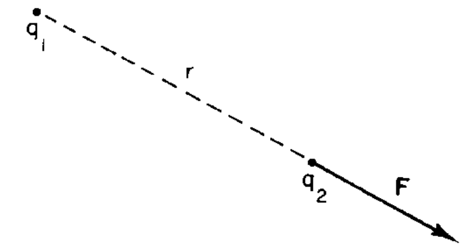
Coloumb's Law, Superposition and Electric Field

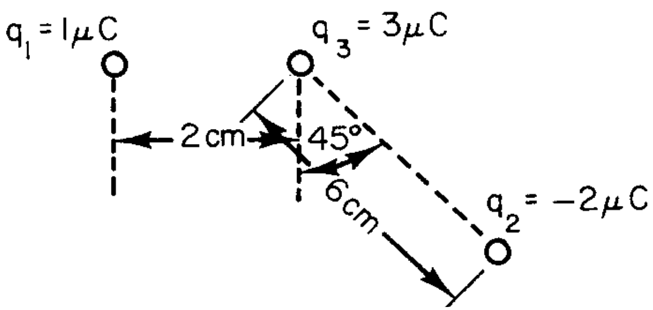
- Electrical force

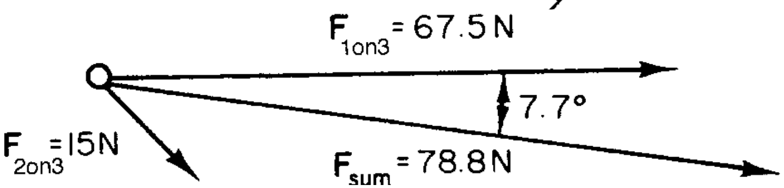
$$|\mathbf{F}| = \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_1 q_2}{r^2}$$

- Superposition
- Electric field

$$|\mathbf{E_1}| = \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{q_1}{r^2}$$



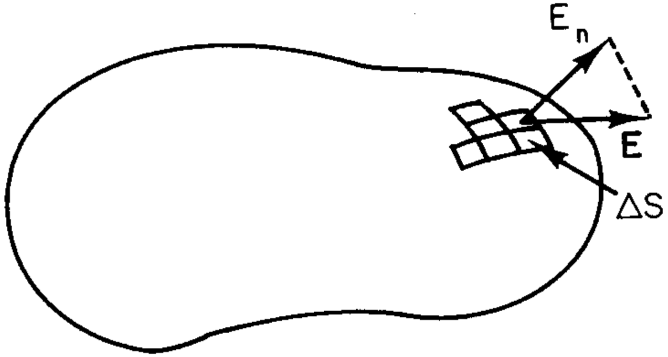




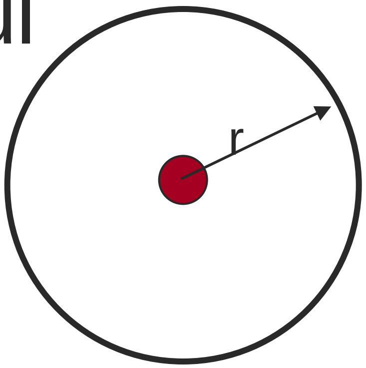
$$\mathbf{F} = q_2\mathbf{E}$$

Gauss's Law

$$\iint_{\text{closed surface}} E_n \, dS = \frac{q}{\varepsilon_o}$$



- Always true but not always useful
- Example: point charge
  - Gaussian surface: sphere

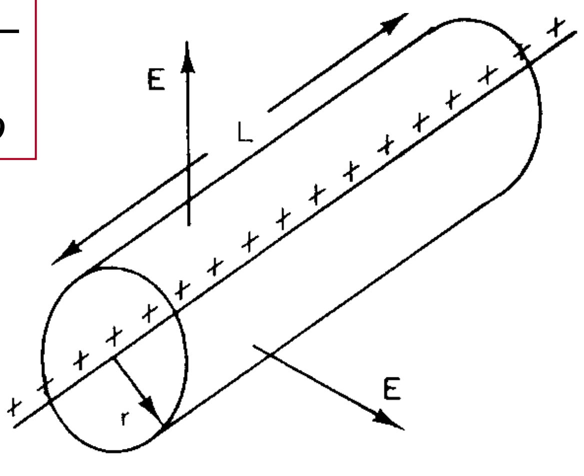


$$\iint E_n \, dS = E \iint dS = E \, 4\pi r^2 = \frac{q}{\varepsilon_o} \Rightarrow E = \frac{q}{4\pi\varepsilon_o r^2}$$

Gauss's Law

- Example: infinitely long line of charge
  - Gaussian surface: cylindrical surface
  - Charge density λ C/m

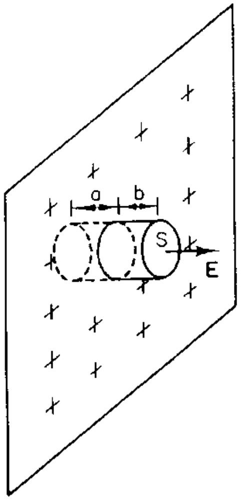
$$E2\pi rL = \frac{\lambda L}{\varepsilon_o} \Rightarrow E = \frac{\lambda}{2\pi r\varepsilon_o}$$

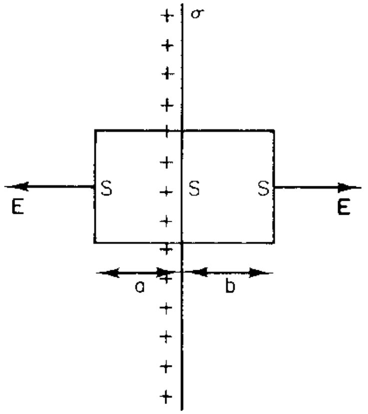


Gauss's Law

- Example: Sheet of charge
  - Gaussian surface: cylinder
  - Charge density: σ C/m^2

$$E(2S) = \frac{\sigma S}{\varepsilon_o} \Rightarrow E = \frac{\sigma}{2\varepsilon_o}$$

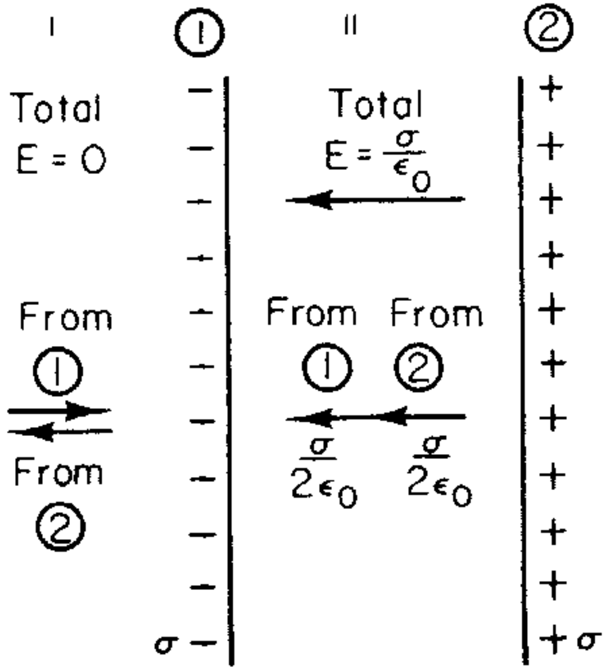




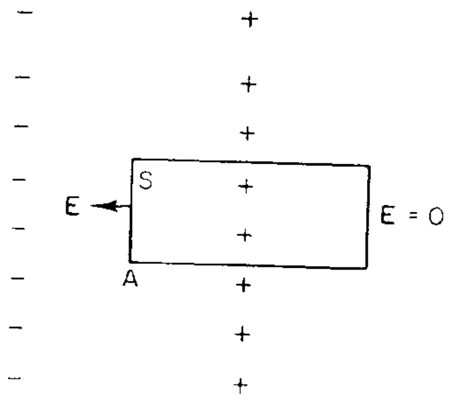
Gauss's Law

- Example:

  Two infinite sheets of charge



- Cell membrane



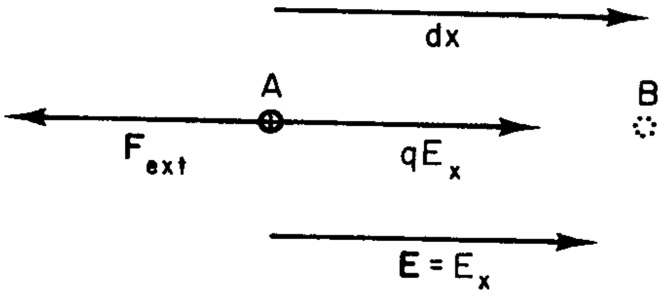
Potential Difference

- Potential energy difference per unit charge

$$\Delta v = -\int_A^B E_x dx$$

$$E_x = -\frac{\partial v}{\partial x}$$

$$v(B) = -\int_\infty^B E_x dx$$

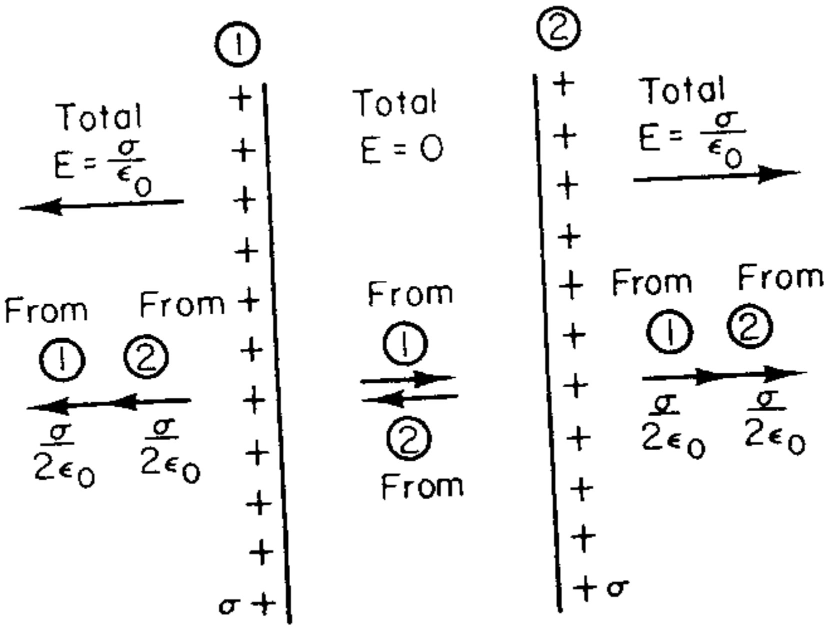


Conductors

- Electric charges are free to move
  - No electric field inside
  - No work is required to move charges
  - Same potential

    if charges are not

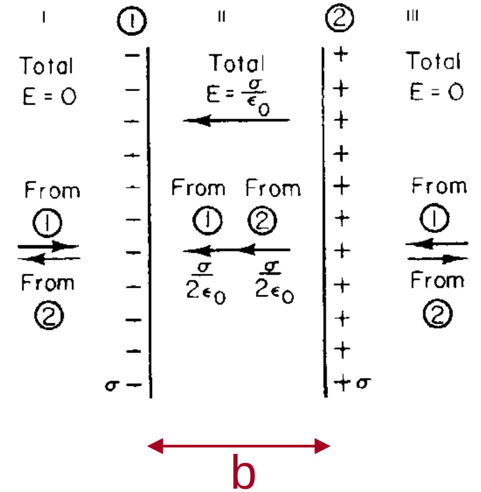
    moving



Capacitance

- Capacitance C (F)

$$Q = Cv$$



$$v = -Eb = \sigma b / \varepsilon_o$$

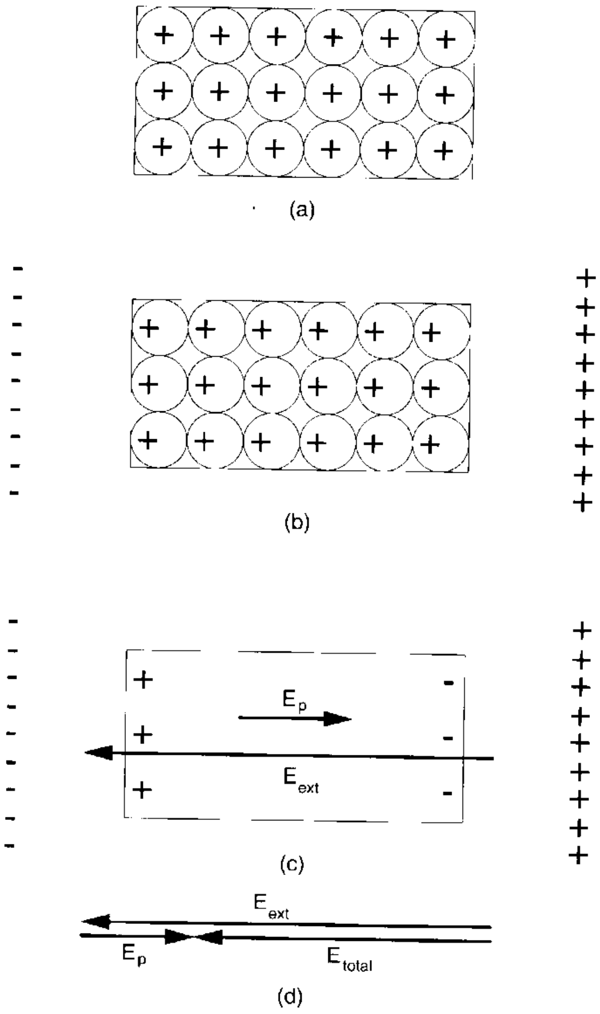
$$C = \frac{Q}{v} = \frac{\sigma S \varepsilon_o}{\sigma b} = \frac{\varepsilon_o S}{b}$$

Dielectrics

- Charges not free to move
  - Polarization field only
  - Partial cancellation inside

$$E_p = -\chi E_{tot}$$

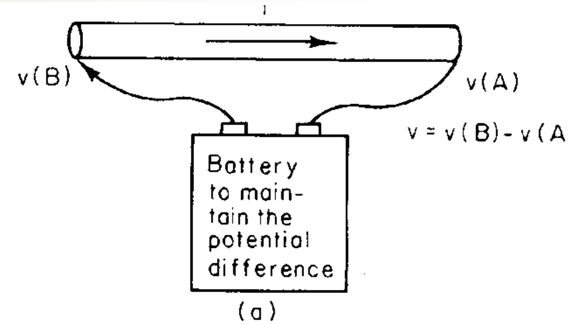
$$E_{tot} = E_{ext} - E_p = \frac{1}{1+\chi} E_{ext} = \frac{1}{\kappa} E_{ext}$$



Current and Ohm's Law

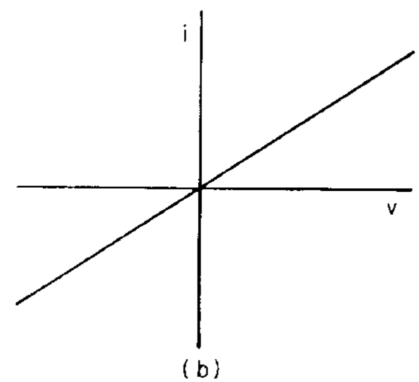
- Ohm's law

$$v = Ri \Leftrightarrow i = Gv$$



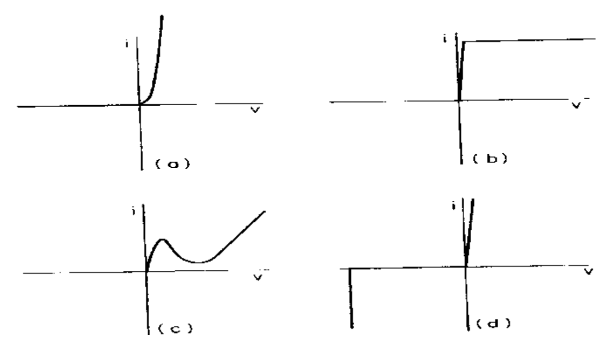
$$\mathbf{j} = \sigma \mathbf{E} = \frac{1}{\rho}\mathbf{E}$$

$$R = \frac{\rho L}{S}$$



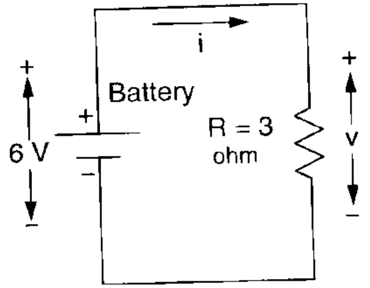
- Power

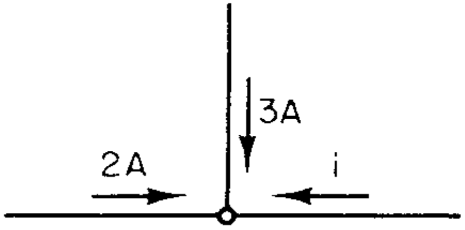
$$P = i^2 R = \frac{v^2}{R}$$

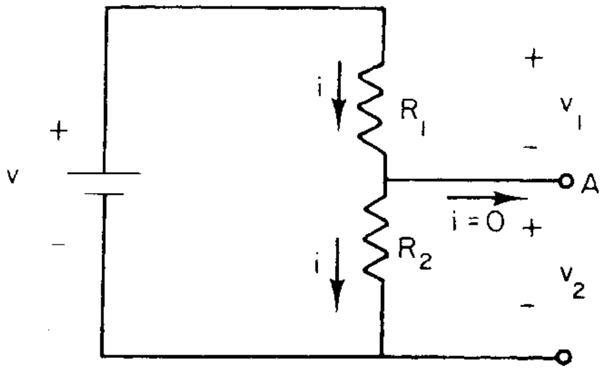


Application of Ohm's Law to Simple Circuits

- Kirchhoff's first law
  - Conservation of charge
- Kirchhoff's second law
  - Conservation of energy

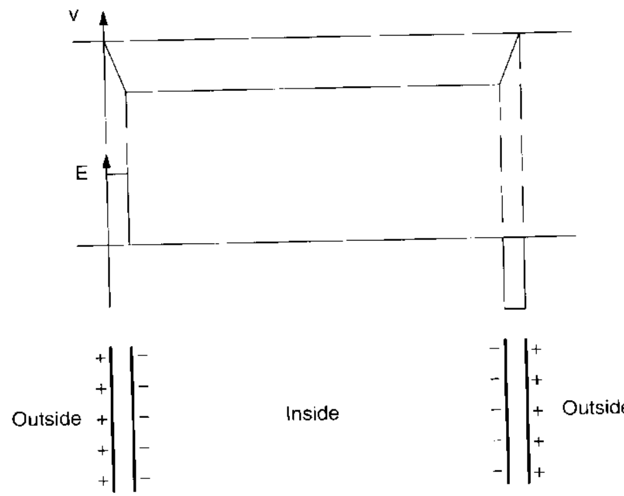


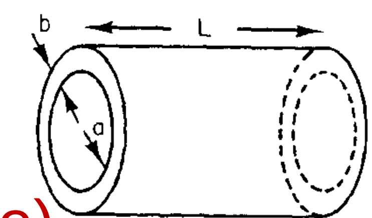


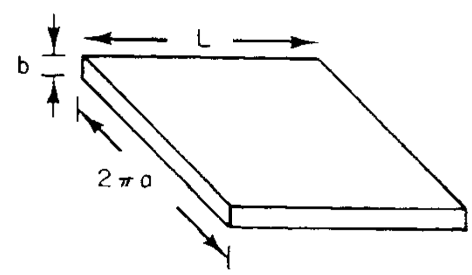


Charge Distribution in Resting Nerve Cell

- Membrane potential -70mV
- Nernst potential
  - Na 30mV, K -90mV, Cl -70
  - Permeability ??
- Membrane capacitance
  - κ=7,
  - b=6nm (mye), 2000nm (unmye)
  - 1μF/cm^2 (mye), lower by 300 (unmye)
  - σ= 700 μC/m^2

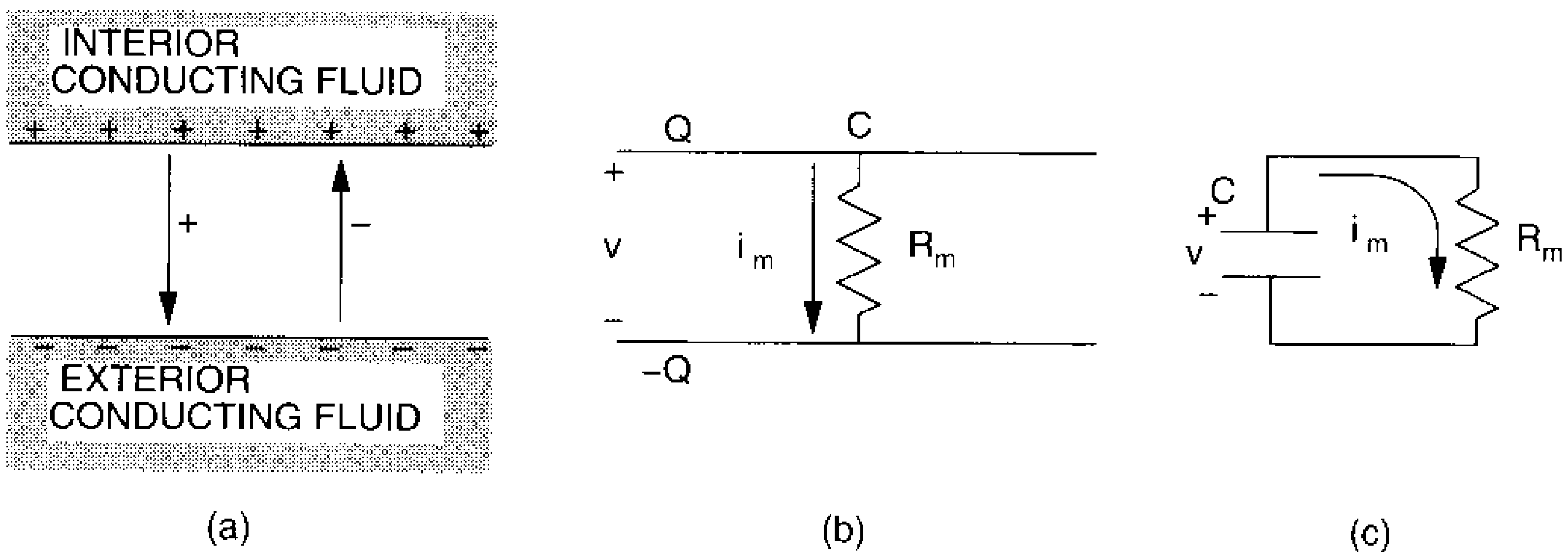






Cable Model for an Axon

- Need to model the complicated flow of charge between inside and outside
- Model a small segment of an axon



Cable Model for an Axon

- Assume no current along the axon

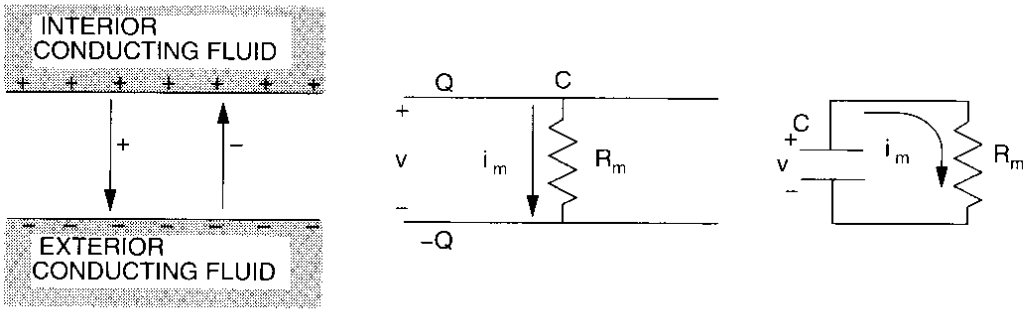
$$-i_m = \frac{dQ}{dt} = C_m \frac{dv}{dt}$$

$$i_m = \frac{v}{R_m}$$

$$\frac{dv}{dt} = -\frac{1}{R_m C_m} v$$

$$v(t) = v_o e^{-t/\tau}$$

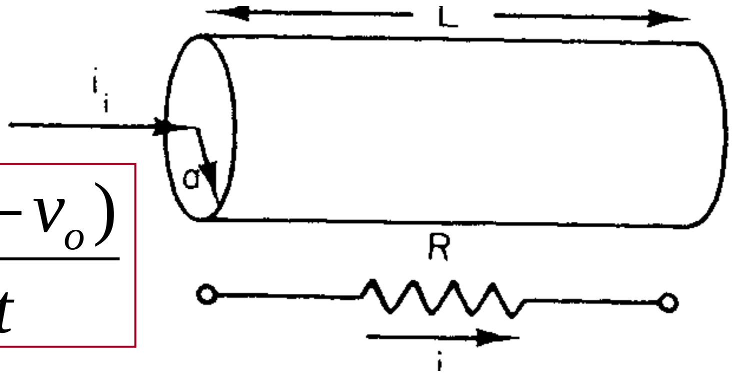
$$\tau = R_m C_m = \frac{\rho_m b}{S} \frac{\kappa \varepsilon_o S}{b} = \kappa \varepsilon_o \rho_m$$



(a) (b) (c)

Cable Model for an Axon

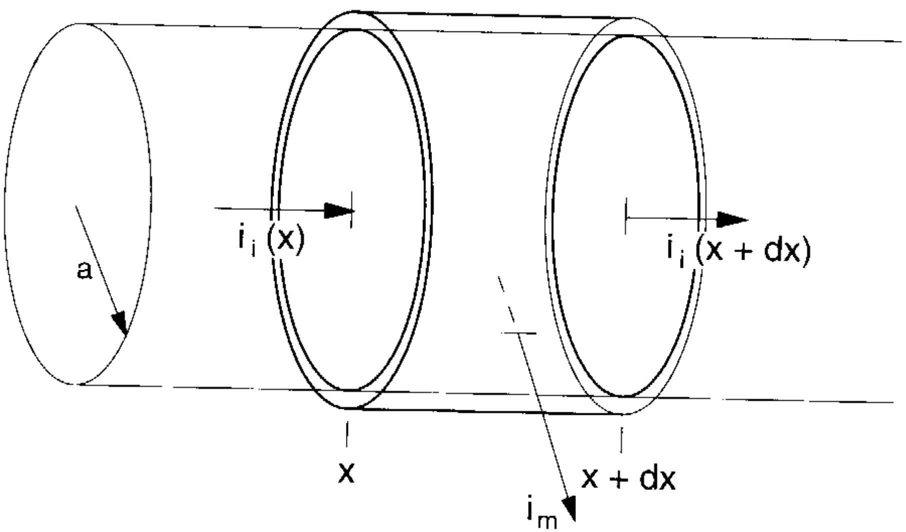
- Consider both longitudinal and membrane currents



$$i_i(x) - i_i(x+dx) - i_m = \frac{dQ}{dt} = C_m \frac{d(v_i - v_o)}{dt}$$

$$-di_i = C_m \frac{dv}{dt} + i_m$$

$$i_i(x) = \frac{v_i(x) - v_i(x+dx)}{r_i dx} = -\frac{1}{r_i}\frac{dv_i}{dx}$$



Cable Model for an Axon

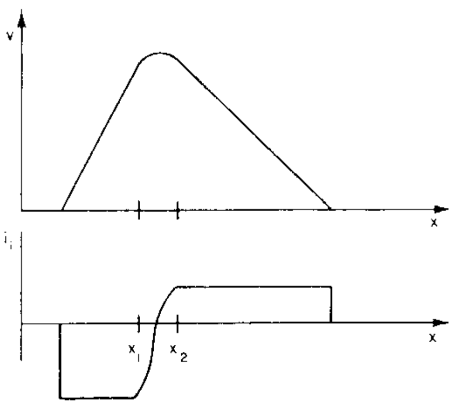
- Dividing by area S= *2 π a dx*

$$-\frac{1}{2\pi a}\frac{di_i}{dx} = c_m \frac{dv}{dt} + j_m$$

- By substitution, *Cable Equation*

$$c_m \frac{\partial v}{\partial t} = -j_m + \frac{1}{2\pi a r_i}\frac{\partial^2 v}{dx^2}$$

  - Similarity to Fick's second law



Electrotonus or Passive Spread

- Membrane assumed ohmic
  - Valid for small changes

$$j_m = g_m (v - v_r)$$

- Substitute into Cable Equation

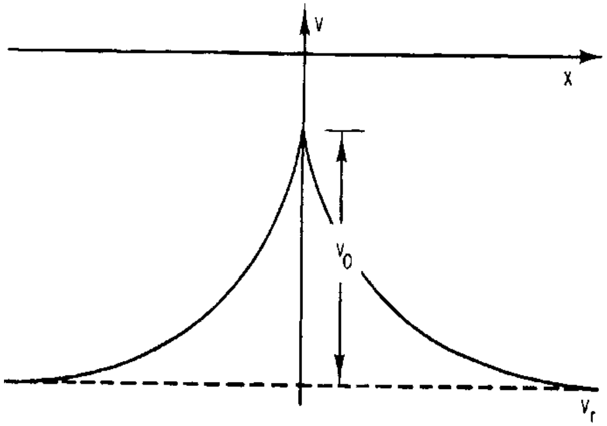
$$\frac{1}{2\pi a r_i g_m}\frac{\partial^2 v}{\partial x^2} - v - \frac{c_m}{g_m}\frac{dv}{dt} = -v_r \Rightarrow \lambda^2 \frac{\partial^2 v}{\partial x^2} - v - \tau\frac{dv}{dt} = -v_r$$

Electrotonus or Passive Spread

- Special case 1: $c_m=0$

$$\lambda^2 \frac{\partial^2 v}{\partial x^2} - v = -v_r$$

$$v - v_r = \begin{cases} v_o e^{-x/\lambda}, & x > 0 \\ v_o e^{x/\lambda}, & x < 0 \end{cases}$$

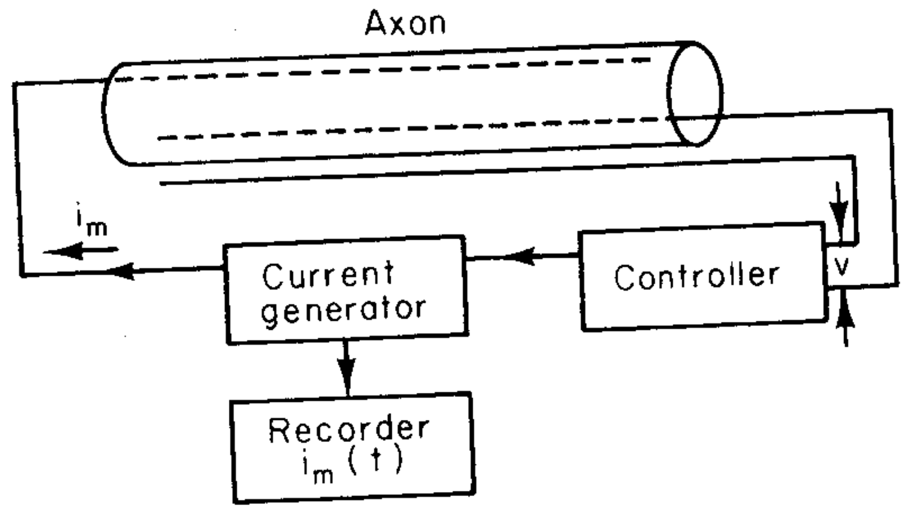


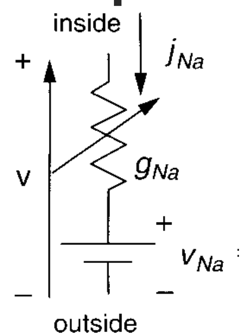
- Special case 2: no dependence on x

$$\tau \frac{dv}{dt} = -(v - v_r) \quad \Rightarrow \quad v - v_r = v_o e^{-t/\tau}$$

Hudgkin-Huxley Model for Membrane Current

- Space-clamped
- Voltage-Clamped





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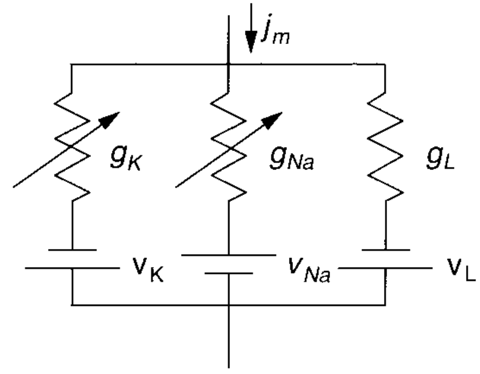


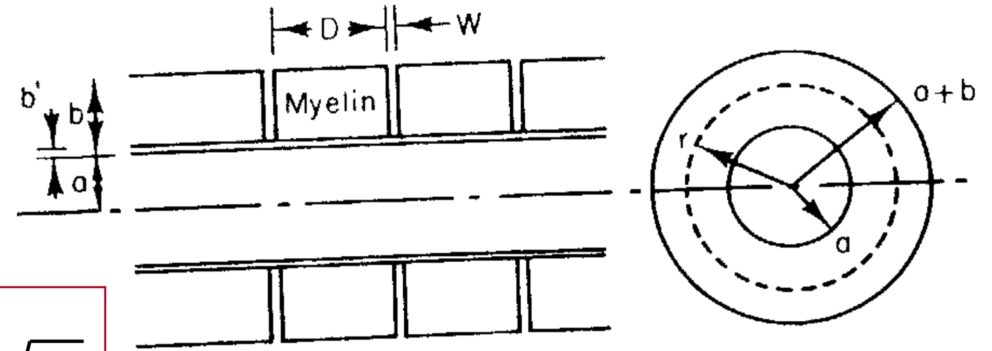
(b)

Myelinated Fibers and Saltatory Conduction

TABLE 6.2. Properties of unmyelinated and myelinated axons of the same radius.

Quantity	Unmyelinated	Myelinated
Axon inner radius, a	5 μm	5 μm
Membrane thickness, b'	6 nm	
Myelin thickness, b		3.4 μm
$\kappa\epsilon_0$	6.20×10^{-11} s^{-1} Ω^{-1} m^{-1}	6.20×10^{-11} s^{-1} Ω^{-1} m^{-1}
Axoplasm resistivity ρ_i	1.1 Ω m	1.1 Ω m
Membrane (resting) or myelin resistivity ρ_m	10^7 Ω m	10^7 Ω m
Time constant $\tau = \kappa\epsilon_0\rho_m$	6.2×10^{-4} s	6.2×10^{-4} s
Space constant λ	$\lambda = \sqrt{\frac{ab\rho_m}{2\rho_i}}$ $= 0.165\sqrt{a}$ $= 370\ \mu\text{m}$	$\lambda = \sqrt{\frac{ab\rho_m}{2\rho_i}} = \sqrt{\frac{0.67a^2\rho_m}{2\rho_i}}$ $= a\sqrt{\frac{0.67\rho_m}{2\rho_i}}$ $= 1750a$ $= 8.8$ mm
Node spacing D		$D = 340a = 1.7$ mm
Conduction speed from model	$u_{\text{unmyelinated}} \propto \lambda/\tau \approx 270\sqrt{a}$	$u_{\text{myelinated}} \propto$ $\lambda/\tau \approx 2.9 \times 10^6 a$ or $D/\tau = 0.55 \times 10^6 a$
Conduction speed, empirical	$u_{\text{unmyelinated}} \approx 1800\sqrt{a}$	$u_{\text{myelinated}} \approx 17 \times 10^6 a$
Ratio of empirical to model conduction speed	6.7	5.9 or 31
Space constant using thick membrane model		$\lambda = a\sqrt{\frac{\ln(1+b/a)\rho_m}{2\rho_i}}$ $= a\sqrt{\frac{\ln(1.67)\rho_m}{2\rho_i}}$ $= 1530a$ $= 7.6$ mm

Myelinated Fibers and Saltatory Conduction



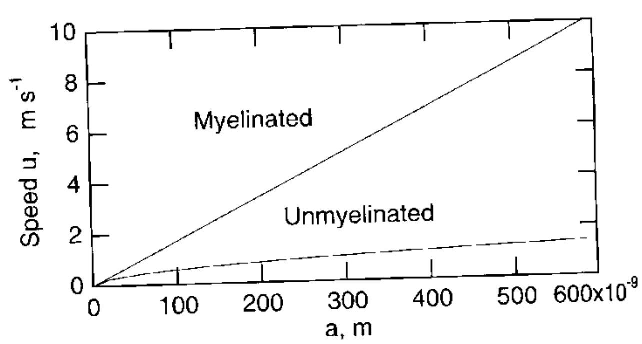
- Unmyelinated
  - Electrotonus

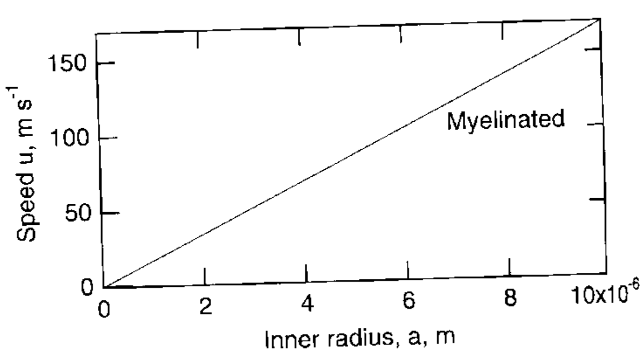
$$u_{unmyelinated} \propto \frac{\lambda}{\tau} \approx 270\sqrt{a}$$

- Myelinated

$$u_{myelinated} \propto \frac{D}{\tau} \approx 0.55 \times 10^6 a$$

- Empirical values agree





Problem Assignment

- Problems 1, 2, 3, 5, 6, 10, 12, 13, 18, 19, 20, 21, 22, 24, 25, 27, 31, 32, 60