Ultrasound Bioinstrumentation

Topic 1

Introduction to Scalar Diffraction Theory

CAIRO UNIVERSITY

Analysis of 2D Signals and Systems

- Basic concepts
  - Linear systems
  - Space invariance
  - Linear transformations
  - Fourier analysis
  - Sampling

Fourier Transform

- Forward transform (*Analysis*)

$$\mathcal{F}\{g\} = \iint_{-\infty}^{\infty} g(x, y) \exp[-j2\pi(f_X x + f_Y y)]\, dx\, dy.$$

- Inverse transform (*Synthesis*)

$$\mathcal{F}^{-1}\{G\} = \iint_{-\infty}^{\infty} G(f_X, f_Y) \exp[j2\pi(f_X x + f_Y y)]\, df_X\, df_Y.$$

Existence of Fourier Transform

- Sufficient (not necessary) conditions:
  - g absolute-integrable
  - g has finite discontinuities/max/min
  - g has no infinite discontinuities
- Bracewell: “physical possibility is a valid sufficient condition for the existence of a transform”
  - Example: dirac-delta function

Fourier Transform as a Decomposition: 1D Case

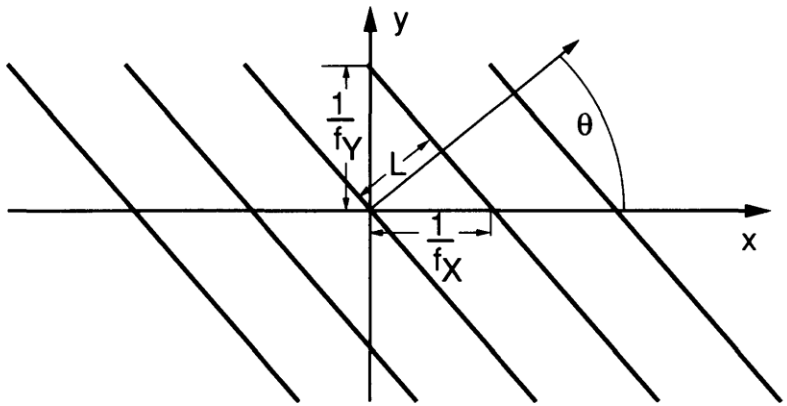
- Linearity enables decomposition into sum of elementary functions
- Fourier analysis is an example of such decomposition:

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi f t)\,df$$

  - Physical realization: pure harmonics
  - Weighting functions: complex G values

Fourier Transform as a Decomposition: 2D Case

- Elementary functions: 2D harmonics
- Frequency “pair”
  - Physical realization: plane waves
  - Spatial period: distance between zero phase lines (wavefronts)



$$L = \frac{1}{\sqrt{f_X^2 + f_Y^2}}.$$

Fourier Transform Theorems

- Linearity theorem

$$\mathcal{F}\{\alpha g + \beta h\} = \alpha \mathcal{F}\{g\} + \beta \mathcal{F}\{h\}$$

- Similarity theorem

$$\mathcal{F}\{g(ax, by)\} = \frac{1}{|ab|} G\left(\frac{f_X}{a}, \frac{f_Y}{b}\right)$$

Fourier Transform Theorems

- Shift theorem

$$\mathcal{F}\{g(x-a, y-b)\} = G(f_X, f_Y)\exp[-j2\pi(f_X a + f_Y b)]$$

- Parseval's theorem

$$\iint_{-\infty}^{\infty} |g(x, y)|^2 \, dx\, dy = \iint_{-\infty}^{\infty} |G(f_X, f_Y)|^2 \, df_X\, df_Y$$

Fourier Transform Theorems

- Convolution theorem

$$\mathcal{F}\left\{\iint_{-\infty}^{\infty} g(\xi,\eta)\, h(x-\xi, y-\eta)\, d\xi\, d\eta\right\} = G(f_X, f_Y)\, H(f_X, f_Y)$$

- Autocorrelation theorem

$$\mathcal{F}\left\{\iint_{-\infty}^{\infty} g(\xi,\eta)\, g^*(\xi - x, \eta - y)\, d\xi\, d\eta\right\} = |G(f_X, f_Y)|^2$$

$$\mathcal{F}\{|g(x,y)|^2\} = \iint_{-\infty}^{\infty} G(\xi,\eta)\, G^*(\xi - f_X, \eta - f_Y)\, d\xi\, d\eta$$

Fourier Transform Theorems

- Fourier integral theorem

$$\mathcal{F}\mathcal{F}^{-1}\{g(x, y)\} = \mathcal{F}^{-1}\mathcal{F}\{g(x, y)\} = g(x, y)$$

Separable Functions

- Rectangular coordinates

$$g(x, y) = g_X(x)\, g_Y(y)$$

- Polar coordinates

$$g(r, \theta) = g_R(r)\, g_\Theta(\theta)$$

Fourier Analysis of Separable Functions

- Rectangular
  - Calculation of 2D transform in terms of two 1D transforms

$$\mathcal{F}\{g(x,y)\} = \iint_{-\infty}^{\infty} g(x,y)\exp[-j2\pi(f_X x + f_Y y)]\,dx\,dy$$

$$= \int_{-\infty}^{\infty} g_X(x)\exp[-j2\pi f_X x]\,dx \int_{-\infty}^{\infty} g_Y(y)\exp[-j2\pi f_Y y]\,dy$$

$$= \mathcal{F}_X\{g_X\}\mathcal{F}_Y\{g_Y\}.$$

Fourier Analysis of Separable Functions

- Polar
  - Not as simple
  - Useful cases: circularly symmetric functions

$$g(r,\theta) = g_R(r)$$

  - Fourier-Bessel transform or Hankel transform of 0th order

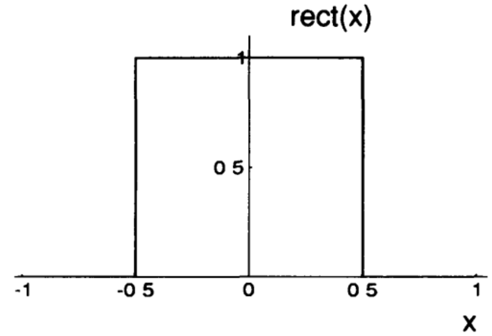
$$G_o(\rho,\phi) = G_o(\rho) = 2\pi \int_0^\infty r g_R(r) J_0(2\pi r\rho)\, dr$$

$$g_R(r) = 2\pi \int_0^\infty \rho G_o(\rho) J_0(2\pi r\rho)\, d\rho$$

Useful Functions

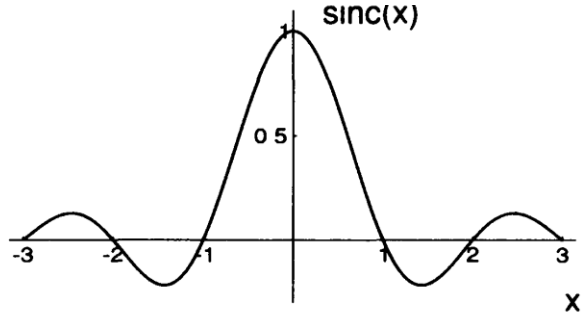
Rectangle function

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



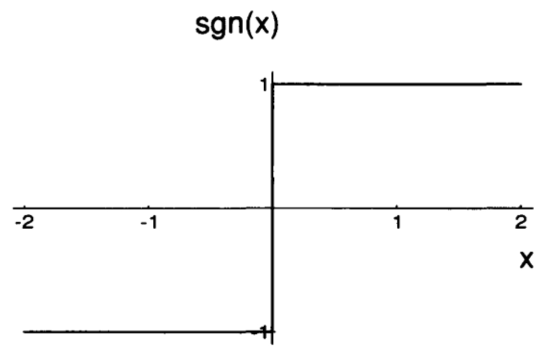
Sinc function

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



Signum function

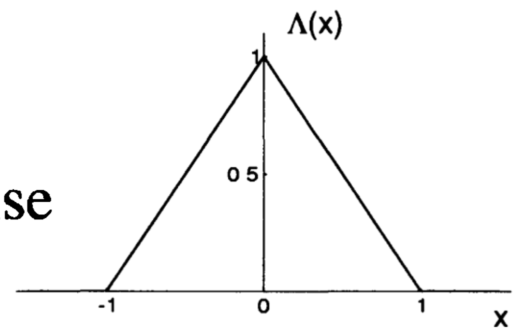
$$\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$



Useful Functions

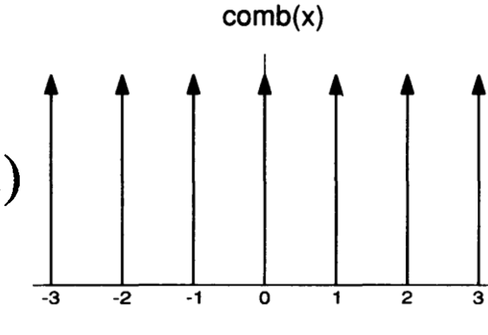
Triangle function

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$



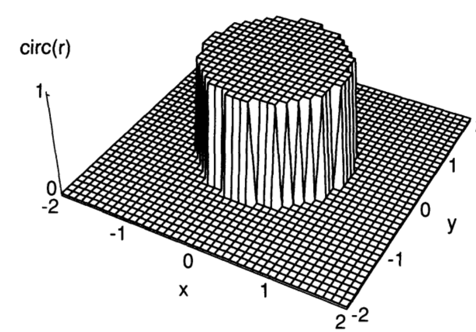
Comb function

$$\text{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$$



Circle function

$$\text{circ}(\sqrt{x^2 + y^2}) = \begin{cases} 1 & \sqrt{x^2 + y^2} < 1 \\ \frac{1}{2} & \sqrt{x^2 + y^2} = 1 \\ 0 & \text{otherwise.} \end{cases}$$



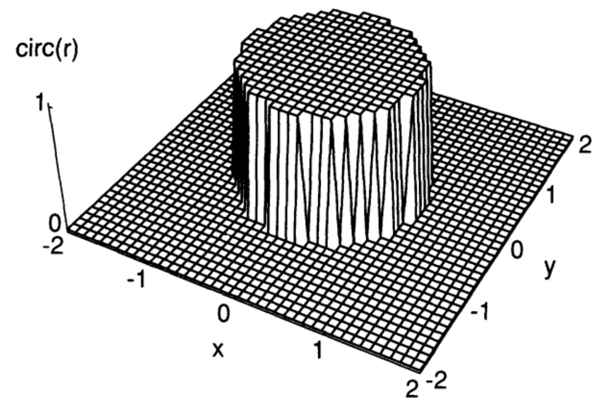
Fourier Transform Pairs

Transform pairs for some functions separable in rectangular coordinates.

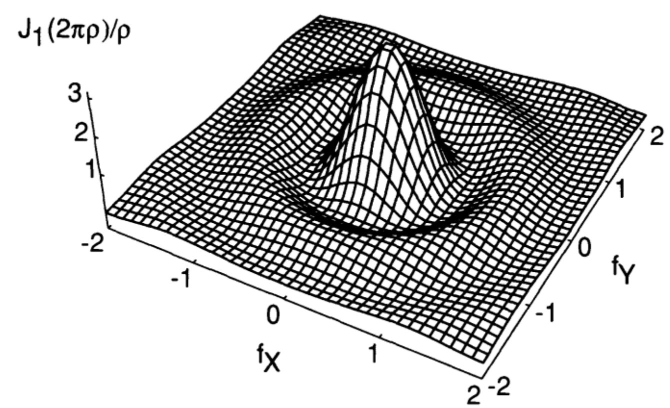
Function	Transform
$\exp[-\pi(a^2x^2 + b^2y^2)]$	$\frac{1}{\lvert ab\rvert} \exp\left[-\pi\left(\frac{f_X^2}{a^2} + \frac{f_Y^2}{b^2}\right)\right]$
$\mathrm{rect}(ax)\,\mathrm{rect}(by)$	$\frac{1}{\lvert ab\rvert} \mathrm{sinc}(f_X/a)\,\mathrm{sinc}(f_Y/b)$
$\Lambda(ax)\,\Lambda(by)$	$\frac{1}{\lvert ab\rvert} \mathrm{sinc}^2(f_X/a)\,\mathrm{sinc}^2(f_Y/b)$
$\delta(ax, by)$	$\frac{1}{\lvert ab\rvert}$
$\exp[j\pi(ax + by)]$	$\delta(f_X - a/2, f_Y - b/2)$
$\mathrm{sgn}(ax)\,\mathrm{sgn}(by)$	$\frac{ab}{\lvert ab\rvert} \frac{1}{j\pi f_X} \frac{1}{j\pi f_Y}$
$\mathrm{comb}(ax)\,\mathrm{comb}(by)$	$\frac{1}{\lvert ab\rvert} \mathrm{comb}(f_X/a)\,\mathrm{comb}(f_Y/b)$
$\exp[j\pi(a^2x^2 + b^2y^2)]$	$\frac{j}{\lvert ab\rvert} \exp\left[-j\pi\left(\frac{f_X^2}{a^2} + \frac{f_Y^2}{b^2}\right)\right]$
$\exp[-(a\lvert x\rvert + b\lvert y\rvert)]$	$\frac{1}{\lvert ab\rvert} \frac{2}{1 + (2\pi f_X/a)^2} \frac{2}{1 + (2\pi f_Y/b)^2}$

Fourier-Bessel Example Pair

$$\mathrm{circ}(r) = \begin{cases} 1 & r < 1 \\ \frac{1}{2} & r = 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\mathcal{B}\{\mathrm{circ}(r)\} = \frac{1}{2\pi\rho^2}\int_0^{2\pi\rho} r' J_0(r')\, dr' = \frac{J_1(2\pi\rho)}{\rho}$$



Local Spatial Frequency

- General function

$$g(x, y) = a(x, y)\exp[j\phi(x, y)]$$

- Local spatial frequency pair defined as:

$$f_{lX} = \frac{1}{2\pi}\frac{\partial}{\partial x}\phi(x, y) \qquad f_{lY} = \frac{1}{2\pi}\frac{\partial}{\partial y}\phi(x, y)$$

- Example: $g(x, y) = \exp[j2\pi(f_X x + f_Y y)]$

$$f_{lX} = \frac{1}{2\pi}\frac{\partial}{\partial x}\left[2\pi(f_X x + f_Y y)\right] = f_X$$

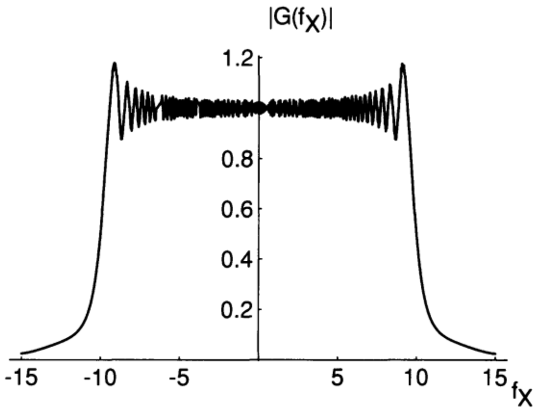
$$f_{lY} = \frac{1}{2\pi}\frac{\partial}{\partial y}\left[2\pi(f_X x + f_Y y)\right] = f_Y.$$

Local Spatial Frequency

- Example: finite chirp
  - Local frequencies = 0 when magnitude=0

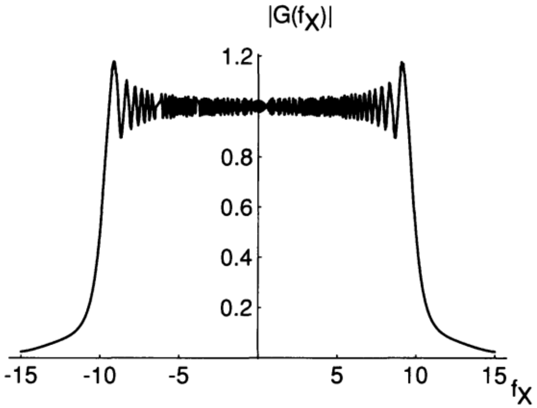
$$g(x,y) = \exp[j\pi\beta(x^2+y^2)]\,\mathrm{rect}\left(\frac{x}{2L_X}\right)\mathrm{rect}\left(\frac{y}{2L_Y}\right)$$

$$f_{lX} = \beta x\,\mathrm{rect}\left(\frac{x}{2L_X}\right) \qquad f_{lY} = \beta y\,\mathrm{rect}\left(\frac{y}{2L_Y}\right)$$



Space-Frequency Localization

- Since the local spatial frequencies are bounded to covering a rectangle of dimensions $2L_x$ X $2L_y$, we conclude that the Fourier spectrum also limited to same rectangular region.
  - In fact this is approximately true, but not exactly so.



The spectrum of the finite chirp function, $L_X = 10, \beta = 1$.

Linear Systems

- A convenient representation of a system is a mathematical operator S{ }, which we imagine to operate on input functions to produce output functions:

$$g_2(x_2, y_2) = \mathcal{S}\{g_1(x_1, y_1)\}$$

- Linear systems satisfy superposition

$$\mathcal{S}\{ap(x_1, y_1) + bq(x_1, y_1)\} = a\mathcal{S}\{p(x_1, y_1)\} + b\mathcal{S}\{q(x_1, y_1)\}$$

Linear Systems: Impulse Response

$$g_1(x_1, y_1) = \iint_{-\infty}^{\infty} g_1(\xi, \eta)\, \delta(x_1 - \xi, y_1 - \eta)\, d\xi\, d\eta$$

$$g_2(x_2, y_2) = \mathcal{S}\left\{ \iint_{-\infty}^{\infty} g_1(\xi, \eta)\, \delta(x_1 - \xi, y_1 - \eta)\, d\xi\, d\eta \right\}$$

$$g_2(x_2, y_2) = \iint_{-\infty}^{\infty} g_1(\xi, \eta)\, \mathcal{S}\{\delta(x_1 - \xi, y_1 - \eta)\}\, d\xi\, d\eta$$

Define: $h(x_2, y_2; \xi, \eta) = \mathcal{S}\{\delta(x_1 - \xi, y_1 - \eta)\}$. Impulse response

Then, $g_2(x_2, y_2) = \iint_{-\infty}^{\infty} g_1(\xi, \eta)\, h(x_2, y_2; \xi, \eta)\, d\xi\, d\eta$

Spatial Invariance: Transfer Function

- A linear imaging system is space-invariant if its impulse response depends only on the x and y distances between the excitation point and the response point such that:

$$h(x_2, y_2; \xi, \eta) = h(x_2 - \xi, y_2 - \eta).$$

$$g_2(x_2, y_2) = \iint_{-\infty}^{\infty} g_1(\xi, \eta)\, h(x_2 - \xi, y_2 - \eta)\, d\xi\, d\eta$$

$$g_2 = g_1 \otimes h \qquad G_2(f_X, f_Y) = H(f_X, f_Y)\, G_1(f_X, f_Y)$$

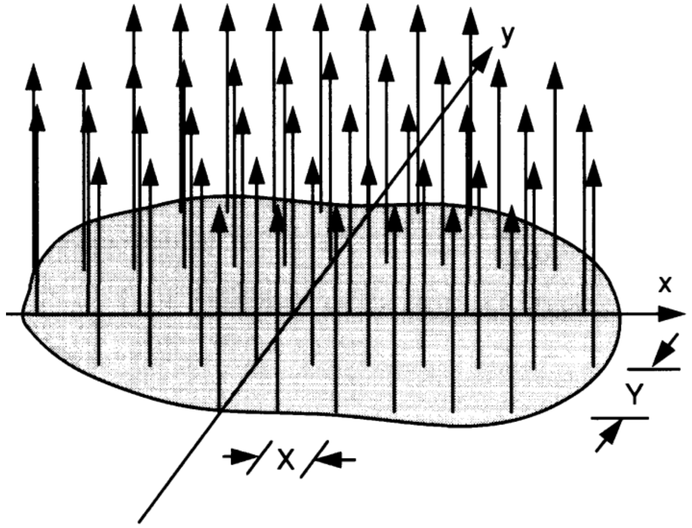
Fourier Transform as Eigendecomposition

- Eigenfunction
  - Function that retains its original form up to a multiplicative complex constant after passage through a system
  - Complex-exponential functions are the eigenfunctions of linear, invariant systems.
- Eigenvalue
  - Weighting applied by the system to an eigenfunction input

Whittaker-Shannon Sampling Theorem

- Sampling

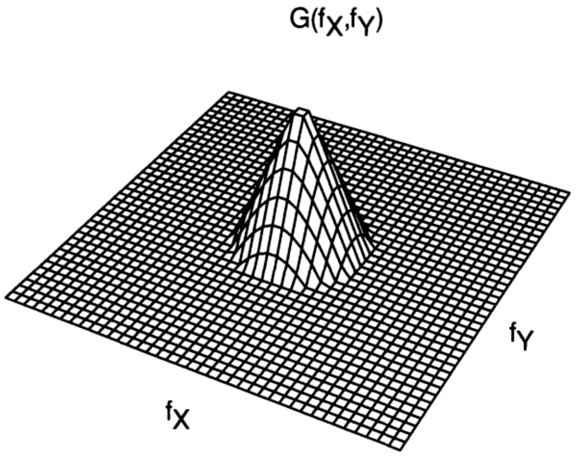
$$g_s(x, y) = \mathrm{comb}\left(\frac{x}{X}\right)\mathrm{comb}\left(\frac{y}{Y}\right)g(x, y).$$

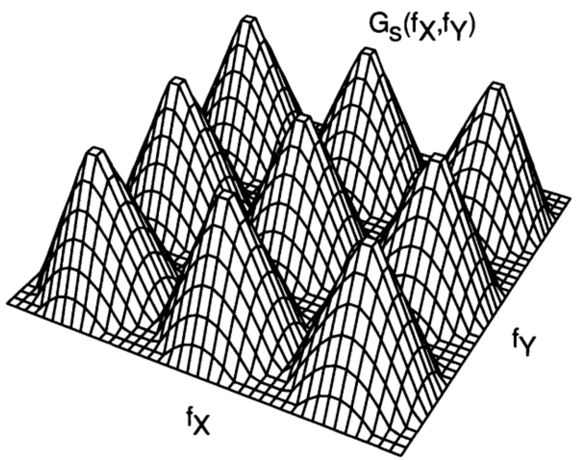


Whittaker-Shannon Sampling Theorem

- Spectrum

$$G_s(f_X, f_Y) = \mathcal{F}\left\{ \mathrm{comb}\left(\frac{x}{X}\right) \mathrm{comb}\left(\frac{y}{Y}\right) \right\} \otimes G(f_X, f_Y)$$





$$G_s(f_X, f_Y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G\left(f_X - \frac{n}{X}, f_Y - \frac{m}{Y}\right).$$

Whittaker-Shannon Sampling Theorem

- Exact recovery of a bandlimited function can be achieved from an appropriately spaced rectangular array of its sampled values

$$X \le \frac{1}{2B_X} \qquad \text{and} \qquad Y \le \frac{1}{2B_Y}.$$

$$g(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} g\left(\frac{n}{2B_X}, \frac{m}{2B_Y}\right) \operatorname{sinc}\left[2B_X\left(x - \frac{n}{2B_X}\right)\right] \operatorname{sinc}\left[2B_Y\left(y - \frac{m}{2B_Y}\right)\right]$$

Space-Bandwidth Product

- Measure of complexity
  - Quality of optical system

$$M = 16 L_X L_Y B_X B_Y$$

- Has an upper bound for Gaussian functions = $4\pi^2$

Problem Assignments

- Problems: 2.1, 2.6, 2.10, 2.11, 2.13